Summing Binomial Coefficients

ORMC

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1 Basic Binomial Sums

Problem 1.1. Find $\sum_{i=a}^{b} \binom{i}{2}$, using a telescoping sum.

Problem 1.2. Prove that

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

2 Problems from "Problem Solving Through Problems"

Problem 2.1. Sum

$$\sum_{j=0}^{n} \sum_{i=j}^{n} \binom{n}{i} \binom{i}{j}.$$

Problem 2.2. Show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Problem 2.3. Use binomial sums to find a formula for $\sum_{k=0}^{n} k^3$.

3 Problems from "Putnam and Beyond"

Problem 3.1. Let F_n be the nth Fibonacci number, with $F_1 = F_2 = 1$. Show that

$$F_1 \binom{n}{1} + F_2 \binom{n}{2} + \cdots + F_n \binom{n}{n} = F_{2n}.$$

Hint: You can use Binet's Formula:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Problem 3.2. Let $a_1, a_2, \ldots$ be an arithmetic sequence - that is, there is some d such that for all n, $a_{n+1} = a_n + d$. Let $S_n = a_1 + a_2 + \cdots + a_n$, $n \geq 1$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} a_{k+1} = \frac{2^n}{n+1} S_{n+1}.$$

4 Competition Problems

Problem 4.1 (2017 BAMO Problem 3)**.** Consider the $n \times n$ "multiplication table" below. The numbers in the first column multiplied by the numbers in the first row give the remaining numbers in the table:

1	2	3	$\cdots$	n
2	4	6	$\cdots$	$2n$
3	6	9	$\cdots$	$3n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$3n$	$\cdots$	n^2

We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, in the case $n = 5$, here is one such possible path, with all the numbers along the path circled:

(1)	(2)	(3)	4	5
2	4	(6)	(8)	10
3	6	9	(12)	15
4	8	12	(16)	20
5	10	15	(20)	(25)

If we add up the circled numbers in the example above (including the start and end squares), we get 93. Considering all such possible paths on the $n \times n$ grid:

- What is the smallest sum we can possibly get when we add up the numbers along such a path? Express your answer in terms of n, and prove that it is correct.
- What is the largest sum we can possibly get when we add up the numbers along such a path? Express your answer in terms of n, and prove that it is correct.

Problem 4.2 (2010 USAMO Problem 2)**.** There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \ldots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

Hint: Let s_k be the maximum number of times the student with height h_k can switch forward. How much higher is s_{k+1} than s_k?

Problem 4.3 (2000 Putnam B5)**.** Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows. The integer a is in S_{n+1} if and only if exactly one of $a - 1$ or a is in S_n. Show that there are infinitely many integers N for which

$$S_N = S_0 \cup \{N + a | a \in S_0\}.$$

Problem 4.4 (1992 Putnam B2)**.** For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.)

Problem 4.5 (2003 Putnam B2)**.** Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < 2/n$.