Competition

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1 Rules

You will be split into N teams and the team with the most points at the end will win. Questions that don't specify a number of points yield one point if solved correctly. Wrong answers are not penalized, but a single final answer is required. Problems in a given section are listed in order of increasing difficulty. Some of the questions in later sections require the presence of an instructor, so, manage your time efficiently.

2 Warmup

Before starting the competition we ask for each team to choose an integer between 0 and 100. Let M be the average of the teams' choices. The team that picks a number closest to $\frac{1}{3}$M wins points equal to the number they chose.

3 Algebra

Problem 3.1.
Starting with some gold coins and some empty treasure chests, I tried to put 9 gold coins in each treasure chest, but that left 2 treasure chests empty. So instead I put 6 gold coins in each treasure chest,but then I had 3 gold coins leftover. How many gold coins did I have?

Problem 3.2.
A fifth number, n, is added to the set $\{3, 6, 9, 10\}$ to make the mean of the set of five numbers equal to its median. What is the number of possible values of n?

Problem 3.3.
How many different real numbers x satisfy the equation $(x^2 - 5)^2 = 16$?

Problem 3.4.
If you add the denominator of the fraction $\frac{1}{3}$ to both its numerator and denominator, the fraction will double. Find a fraction that will triple under the same procedure.

Problem 3.5.
At a ballroom dancing competition, a boy always dances with a girl, a pair can dance together a few times, and the number of boys and girls can be different. At the end of the competition, the competitors were asked in random order how many dances each of them has performed. The answers were 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6, 6, 6. Is it possible for everyone to have answered truthfully?

Problem 3.6.
Find the equation of the second degree polynomial passing through $(0,0)$, $(1,1)$, and $(3,-3)$.

Problem 3.7.
A rectangular pool table with one pocket on each corner has side lengths 4 feet and 6 feet. A ball is placed at one corner and aimed at an angle of 45°. How many times will the ball bounce on the walls before it reaches a pocket.

5 points:

Problem 3.8.
How many integers, n, from 1 to 100 have an odd number of divisors (including 1 and n).

4 Counting Combinatorics and Probability

Problem 4.1.
Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the people has at least 2 apples?

Problem 4.2.
How many 7-digit palindromes (numbers that read the same forwards and backwards) that do not have more than 2 occurrences of any digit are there?

Problem 4.3.
From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?

Problem 4.4.
How many positive cubes divide $3! \cdot 5! \cdot 7!$?

Problem 4.5.
Given a chess board in how many ways can you place 8 rooks such that non of them can capture each other.

Problem 4.6.
You start at the bottom left corner of a chess board and each time you can move one step to the right or one step up. In how many ways can you get to the top right corner?

5 points:

Problem 4.7.
Let us call a number "prime-looking" if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers less than 1000 are there?

5 Information Theory

After devising strategies for the problems in this section you will need an instructor present to test them.

Problem 5.1.
An instructor will guess a number from 1 to 100. Your goal is to guess it by asking the fewest possible Yes or No questions. After guessing the number you will get N points where $N = 8-$ number of questions asked.

5 points:

Problem 5.2.
A member of your team will be given a secret sequence of 0s and 1s of some length and will be asked to add a bit of their choice at the end of this sequence. Then an instructor will take this sequence and choose to either flip one, or none of the bits and give the result to another member of the team. Finally, that member will have to guess whether the instructor did or didn't alter the sequence. Devise a strategy and test it out with an instructor. To get 5 points your team will need to guess correctly 3 times in a row.

10 points:

Problem 5.3.
3 members from your teams will be assigned colored hats. Each hat can be either red, blue, or green. (we can have multiple hats of the same color). You can't see the color of your hat but you can see your two teammates' hats. All three of you then try to guess the color of your own hat, and you win if at least one of you gets it right. Devise a strategy that guarantees victory and test it out with an instructor. To get 10 points your team will need to win 4 times in a row.

6 Games

Problem 6.1.
Beat an instructor at the following game to get a point. You can play a maximum of three times for a maximum of 3 points.
You start with a pile of 15 blocks. Players alternate turns and each time can remove 1, 3, or 6 blocks from the pile. The player who removes the last one wins. When playing against an instructor the student will play first.

Problem 6.2.
Play the game of Sim against an instructor, with the instructor playing first. Every victory will get your team 2 points. You can play up to 3 times for a maximum of 6 points.

10 points:

Problem 6.3.
Consider a deck of 6 cards with the values from 1 to 6. Two players each select a random card from the deck without revealing it to the other player. Then each player has the right to ask for a trade, which the other player can choose to accept or decline. After they are done trading (or didn't trade at all) the player with the largest card number wins. Assuming both players play optimally, will a trade ever take place? If yes, what cards do the players need to be dealt for that to happen?

7 Casino

Problem 7.1.
You may choose to roll dice A or dice B 10 times. Each time you roll dice A your score is incremented by the number you roll multiplied by 0.2, unless you roll a 1,2, or a 3, in which case you get 0. With dice B your score is incremented by the number you roll multiplied by 0.2, unless that number is 3 or 4 in which case you get 0. Choose carefully between A and B, and talk to an instructor to verify your score.

Problem 7.2.
An instructor gives you three face down cards, two of which are 2's and the other an Ace. You initially choose a card and then the instructor turns over a 2 from the remaining two cards. Finally you can choose to stick with your original choice or switch to the other remaining card. If your final choice is an Ace you get two points, otherwise 0. You can play this game up to 3 times for a maximum of 6 points.