THE AM-GM INEQUALITY

MATH CIRCLE (ADVANCED) 11/11/2012

The arithmetic mean (AM) and geometric mean (GM) of the numbers $a_1, a_2, \ldots, a_n \geq 0$ are given by

$$AM = \frac{a_1 + \cdots + a_n}{n} \text{ and } GM = \sqrt[n]{a_1 \cdots a_n}.$$

0) a) Calculate the AM and GM of the following sets of numbers:

i) $1, 2, 3, 6$

ii) $0, 4, 8, 20$

iii) $1, 3, 4, 7, 8$

iv) $4, 4, 4, 4$

b) What do you notice about AM compared to GM?

See ↓

The AM-GM inequality states that if $a_1, a_2, \ldots, a_n \geq 0$, then

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \text{ with equality if and only if } a_1 = a_2 = \cdots = a_n.$$

2) a) Write out the AM-GM inequality for the numbers a, b.

b) Prove your statement in a).

$$\frac{a+b}{2} \geq \sqrt{ab} \Leftrightarrow (a+b)^2 \geq 4ab \Leftrightarrow a^2 + 2ab + b^2 \geq 4ab \Leftrightarrow a^2 - 2ab + b^2 \geq 0 \Leftrightarrow (a-b)^2 \geq 0$$

3) Prove the following:

a) $\dfrac{x^2 + y^2}{2} \geq xy$ for any x, y.

Use AM-GM inequality with $a = x^2, b = y^2$.

b) $2(x^2 + y^2) \geq (x+y)^2$ for any x, y.

Use a) and the fact that $2(x^2 + y^2) \geq (x+y)^2 \Leftrightarrow x^2 + y^2 \geq 2xy$.

c) $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{x+y}$ for $x, y \geq 0$.

Hint: Find a common denominator.

4) Prove that

a) $x^2 + y^2 + z^2 \geq xy + yz + zx$ for any x, y, z.

Hint: Use 3a) and add up three inequalities.

b) $(a+b)(a+c)(b+c) \geq 8abc$

Hint: Use the AM-GM inequality three times.

c) $x^2+y^2+1 \geq xy+x+y$ if $x, y \geq 1$, (for a challenge prove it for any x, y!).

Using 3a) we have $x^2+y^2+1 \geq 2xy+1$ so it is enough to prove $2xy+1 \geq xy+x+y$ if $x, y \geq 1$. We have $2xy+1 \geq xy+x+y \Leftrightarrow xy-x-y+1 \geq 0 \Leftrightarrow (x-1)(y-1) \geq 0$.

In general, $x^2+y^2+1-xy-x-y = (x-y)^2+(x-1)^2+(y-1)^2/2$.

d) $x^4+y^4+z^4 \geq xyz(x+y+z)$ for any x, y, z.

Hint: Use 4a) twice.

5) The sum of two non-negative numbers is 10. What is the maximum and minimum value of the sum of their squares?

The sum of the squares is $x^2+(10-x)^2 = 2x^2-20x+100 = 2[(x-5)^2+20]$. The minimum occurs at $x=5$ while the maximum occurs at $x=0$ or $x=10$.

6) Prove the AM-GM inequality for $n=4$.

Using 2a) twice we have:

$$\frac{a+b+c+d}{4} = \frac{1}{2}\left(\frac{a+b}{2}+\frac{c+d}{2}\right) \geq \frac{1}{2}\left(\sqrt{ab}+\sqrt{cd}\right) \geq \sqrt{\sqrt{ab}\sqrt{cd}} = \sqrt[4]{abcd}$$

7) Prove that: (Hint: You may use the AM-GM inequality for any n.)

a) $a^4+b^4+8 \geq 8ab$ for any a, b.

Hint: Apply AM-GM to $x=a^4, y=b^4, z=4, w=4$.

b) $(a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}) \geq 16$ for $a, b, c, d \geq 0$.

Hint: What does AM-GM tell use about the two terms of the left-hand side?

c) $3x^3-6x^2+4 \geq 0$ for $x \geq 0$.

Hint: Apply AM-GM to $a=2x^3, b=x^3, c=4$.

d) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$ for $a, b, c \geq 0$.

Hint: Simply use AM-GM.

8)

a) Prove the AM-GM inequality for $n=3$.

Suppose a, b, c are our three numbers. Applying AM-GM to $x=a, y=b, z=c, w=\sqrt[3]{abc}$ we have

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw} = \sqrt[4]{w^4} = w \Rightarrow x+y+z \geq 3w \Rightarrow \frac{a+b+c}{3} \geq \sqrt[3]{abc}.$$

b)* Prove by induction that the AM-GM inequality holds for n a power of 2.

Hint: Use the same trick as in 6) for the induction step.

c)* Prove the AM-GM inequality for all n.

Hint: Use the same trick as in 8a).

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"