3 Individual Round

1. For each positive integer \( k \), let \( S_k \) denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is \( k \). For example, \( S_3 \) is the sequence 1, 4, 7, 10, \ldots. For how many values of \( k \) does \( S_k \) contain the term 2005?

2. There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

(A) 1    (B) 2    (C) 3    (D) 4    (E) 5

3. Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

(A) 6    (B) 9    (C) 12    (D) 15    (E) 18

4. An integer \( N \) is selected at random in the range \( 1 \leq N \leq 2020 \). What is the probability that the remainder when \( N^{16} \) is divided by 5 is 1?

(A) \( \frac{1}{5} \)    (B) \( \frac{2}{5} \)    (C) \( \frac{3}{5} \)    (D) \( \frac{4}{5} \)    (E) 1

5. Let \( \triangle XOY \) be a right-angled triangle with \( m \angle XOY = 90^\circ \). Let \( M \) and \( N \) be the midpoints of legs \( OX \) and \( OY \), respectively. Given that \( XN = 19 \) and \( YM = 22 \), find \( XY \).

(A) 24    (B) 26    (C) 28    (D) 30    (E) 32

6. In the expansion of

\[(1 + x + x^2 + \cdots + x^{27}) (1 + x + x^2 + \cdots + x^{14})^2,\]

what is the coefficient of \( x^{28} \)?

(A) 195    (B) 196    (C) 224    (D) 378    (E) 405
7. A function $f$ is defined recursively by $f(1) = f(2) = 1$ and 

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

(A) 2016    (B) 2017    (C) 2018    (D) 2019    (E) 2020

8. Triangle $ABC$ has $AB = 13$, $BC = 14$ and $AC = 15$. Let $P$ be the point on $AC$ such that $PC = 10$. There are exactly two points $D$ and $E$ on line $BP$ such that quadrilaterals $ABCD$ and $ABCE$ are trapezoids. What is the distance $DE$?

(A) $\frac{42}{5}$    (B) $6\sqrt{2}$    (C) $\frac{84}{5}$    (D) $12\sqrt{2}$    (E) 18

9. How many ordered pairs $(a, b)$ of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot \text{lcm}(a, b) + 12 \cdot \text{gcd}(a, b),$$

where gcd$(a, b)$ denotes the greatest common divisor of $a$ and $b$, and lcm$(a, b)$ denotes their least common multiple?

(A) 0    (B) 2    (C) 4    (D) 6    (E) 8

10. In $\triangle ABC$ shown in the adjoining figure, $M$ is the midpoint of side $BC$, $AB = 12$ and $AC = 16$. Points $E$ and $F$ are taken on $AC$ and $AB$, respectively, and lines $EF$ and $AM$ intersect at $G$. If $AE = 2AF$ then $\frac{EG}{GF}$ equals

(A) $\frac{3}{2}$    (B) $\frac{4}{3}$    (C) $\frac{5}{4}$    (D) $\frac{6}{5}$    (E) not enough information to solve the problem