Catalan Numbers and Related Combinatorics

ORMC

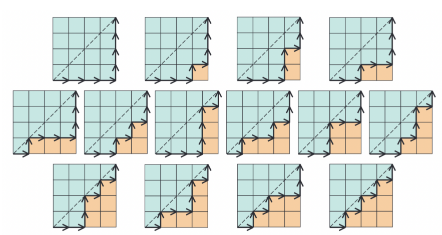
03/03/24

1 Path Counting and Catalan Numbers

Problem 1.1. On a square lattice, how many paths are there that stick to integer points, moving only 1 lattice point at a time, only go up and to the right, never down or left, and go from $(0,0)$ to (m,n)?

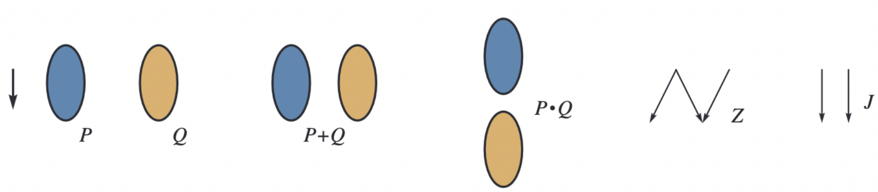
Problem 1.2. How many lattice paths are there, going from $(0,0)$ to (n,n), going right and up without ever going above the line $x=y$?

Here are the valid paths illustrated for $n=4$.



2 Posets

Problem 2.1 (From Igor Pak's Class)**.** For posets $P=(X,\preceq)$ and $Q=(Y,\preceq')$ define the sum $P+Q$ and the product $P\cdot Q$ on the same set $X\cup Y$, as in the figure. Define also a 4-element poset Z as in the figure. Prove that a poset A does not contain Z as an induced subposet (meaning it has no additional relations), if and only if A can be obtained from a single-element poset using the sum and product operations. We call these posets *nice*.

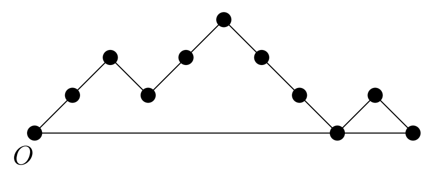


Problem 2.2 (From Igor Pak's Class)**.** Use the previous problem to prove that the number of non-isomorphic posets on n elements which have no induced Z and J as in the figure, is the Catalan number C_n.

3 Competition Problems

Problem 3.1 (1996 USAMO Problem 4)**.** An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

Problem 3.2 (2003 Putnam A5)**.** A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

Problem 3.3 (2005 Putnam A2)**.** Let $\mathbf{S} = \{(a,b) | a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

(i) $p_i \in \mathbf{S}$,

(ii) p_i and p_{i+1} are a unit distance apart, for $1 \le i < 3n$,

(iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1,1)$ and end at $(n,1)$?