Did Dream Cheat?

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This packet is inspired by the YouTube video "How lucky is too lucky?: The Minecraft Speedrunning Dream Controversy Explained" by Stand-up Maths.

Warm-up.

Exercise 1. *A fraction* $\frac{p}{q}$ *has the following properties: (i) it is positive and proper with* $0 < \frac{p}{q} < 1$*, (ii) the numerator and denominator share no common factors, and (iii) the fraction doesn't change if one adds 30 to the numerator and 40 to the denominator. What is the fraction?*

Exercise 2. *Let* $\frac{p_1}{q_1}$ *and* $\frac{p_2}{q_2}$ *be positive proper fractions such that: (i)* p_1 *and* q_1 *have no common factors and neither do* p_2 *and* p_2 *and* q_2*, and (ii)* $q_1 \neq q_2$, $q_1 > 100$ *, and* $q_2 > 100$*. Is it possible that the sum of the fractions* $\frac{p_1}{q_1} + \frac{p_2}{q_2}$ *has a denominator less than 100 (in lowest terms)?*

Exercise 3. *Can the value of a positive fraction increase if we add one to the numerator and one hundred to the denominator?*

Introduction.

Minecraft is a popular computer game in which players mine resources and defeat monsters. Loosely speaking, players win Minecraft by building Nether portals, gathering Ender pearls, locating an End portal, and defeating the final boss: the Ender dragon. Don't worry if you don't know what any of this means; relevant terms will be explained throughout the packet. In recent years, it has been a trend for skilled Minecraft players to "speedrun" Minecraft, which means that they try to beat the game as quickly as possible. At higher levels, the success of Minecraft speedruns are determined to a large degree by luck, since the drop rates of certain key items are lower than others.

Dream is a particularly popular Minecraft YouTuber – he had 15 million subscribers in 2020 and currently has over 30 million subscribers in 2024. In December 2020, Dream posted a Minecraft speedrun time to `speedrun.com`, which was the fourth fastest ever at the time. However, the moderation team of `speedrun.com` was not convinced that Dream was fairly playing the game, and removed his speedrun time from the ranking list. According to the moderators, Dream's acquisition of certain items in the game constituted a significant statistical anomaly, and they concluded that Dream must have been using a modified version of the game. In this packet, you (the astute mathematician) will use probability and statistics to determine whether you think that Dream was, in fact, cheating.

Binomial distribution.

Let's start by simplifying the Minecraft speedrun scenario and see how one might use probability theory to determine whether a sequence of outcomes is "too unlikely" to have occurred. We'll begin by considering some coin tossing experiments; note that we have seen similar experiments in previous worksheets from this year.

Exercise 4. *Suppose I flip four fair coins. List all possible outcomes, and mark the ones where I see at most one head. What is the probability that I see at most one head?*

Exercise 5. *There are four total outcomes when I flip two fair coins (HH, HT, TH, TT). How many total outcomes are there when I flip five coins? How about n coins? Hint: there are two outcomes for each coin.*

Exercise 6. *Recall from previous worksheets that there are* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ *ways to choose a team of* k *players for a team from a pool of* n *unique candidates (read as "*n *choose* k*"), where* $0 \leq k \leq n$ *are integers. Using this formula, when I flip* n *coins, how many ways are there for me to get exactly* k *heads?*

Exercise 7. *Using your answers to the previous two parts, write down a formula for the probability of getting exactly* k *heads when I flip* n *fair coins.*

Now, suppose that I have an unfair coin, which has probability q of landing heads and $1-q$ of landing tails (where $0 \leq q \leq 1$). To guide your intuition, when $q = \frac{1}{4}$, we would expect the coin to land tails three times more often than it lands heads. We'll call this coin $\mathrm{Ber}(q)$ (mathematicians call these things "Bernoulli random variables"). Note that the probabilities of independent coin flips multiply, so $q(1-q)(1-q)$ is the probability of getting one head, then one tail, then one tail.

Exercise 8. *Suppose I flip* $\mathrm{Ber}(q)$ *three times. What is the probability I see exactly zero heads? One? Two? How about three? Verify that when* $q = \frac{1}{2}$*, your answer coincides with your formula from Exercise 4. Also, verify that the four probabilities you find add to one, since exactly one of these outcomes has to occur.*

Exercise 9. *Write down a formula for the probability of seeing exactly* k *heads when* $\mathrm{Ber}(q)$ *is flipped* n *times. Verify that this formula coincides with your answer to Exercise 4 when* $q = \frac{1}{2}$. *Hint: suppose we know the spots in which* k *heads and* $n-k$ *tails occur (for example, this could be* $HHTTTH$ *for* $n = 6$ *and* $k = 3$*). Then, write down the probability of seeing exactly this sequence happen. Multiply this probability by the number of ways in which we can choose* k *spots from a sequence of length* n *to be heads (we derived this in Exercise 3...).*

What you just derived is called the **binomial distribution** $\mathrm{Bin}(n, q)$. In the rest of the packet, we're going to use these ideas to analyze Dream's speedrun.

Hypothesis testing.

Next, we need a way to determine whether an event was too unlikely to have reasonably occurred. In this case, we want to test which of two hypotheses H_0 (called the **null hypothesis**) and H_a (called the **alternative hypothesis**) are true. The way statisticians quantify this is with a **p-value**, which is loosely defined as the probability of seeing an event at least as extreme as the one we've observed, if we assume the null hypothesis to be true.

Fix a number $0 \leq p^* \leq 1$. We accept the null hypothesis if the p-value (called p) has $p > p^*$, and we reject the null hypothesis if $p \leq p^*$. Here, the statistician chooses p^* depending on how much uncertainty they are okay with. A smaller p^* makes it harder to reject the null hypothesis, and a larger p^* makes it easier to reject the null hypothesis. For reference, many scientific experiments use $p^* = 0.05$.

It is okay to use a calculator for the following questions.

Exercise 10. *Suppose I tell you that I flipped a fair coin eleven times and got two heads. Here, H_0 is "the coin is fair" and H_a is "the coin is unfair". How likely is it that I got two or fewer heads, assuming that the coin is fair? In a sense, this probability is a "one-sided" p-value. If $p^* = 0.05$, would you accept or reject the null hypothesis?*

Exercise 11. *Suppose I tell you that I flipped a fair coin eleven times and got two heads. Here, H_0 is "the coin is fair" and H_a is "the coin is unfair". How likely is it that I got two or fewer heads, or that I got two or fewer tails, assuming that the coin is fair? In a sense, this probability is a "two-sided" p-value. If $p^* = 0.05$, would you accept or reject the null hypothesis? Hint: you don't need to do much additional work if you re-use your solution to Exercise 7.*

Exercise 12. *Your good friend Kason (who is known for bragging) tells you that he flipped a fair coin a hundred times and got exactly two heads. Do you believe him? Provide evidence in the form of a p-value. Be as generous as possible (use a "two-sided" hypothesis test as in Exercise 8). What value of p^* would you need to set in order to believe Kason?*

Dream's speedruns.

For the following questions, please use a calculator or online tool (such as `https://stattrek.com/online-calculator/binomial`) to compute the probability that $X \sim \mathrm{Bin}(n, p)$ is less than x. Ask your instructors if you need any help with this. **Important:** please note that the calculator is not meant to handle probabilities that are very close to zero or very close to one, so if you see a probability reported as zero or one, ask an instructor for the exact answer.

Exercise 13. *One phase of a successful Minecraft speedrun is to barter with cleric villagers for an item called an Ender pearl. In an un-modified game of Minecraft, for each transaction with a cleric villager, there is a 4.7% chance that an Ender pearl is given to you. Over the course of 6 livestreams, Dream engaged in 262 barters with cleric villagers and was offered 42 pearl trades. What is the probability that one is offered at least 42 pearl trades in 262 barters?*

Exercise 14. *Another phase of a Minecraft speedrun is to defeat Blazes and obtain Blaze rods. In an un-modified game of Minecraft, each time a Blaze is killed, it has a 50% chance of dropping a Blaze rod. Over the course of their livestreams, Dream killed 305 Blazes and obtained 211 Blaze rods. What is the probability that one obtains at least 211 Blaze rods by killing 305 Blazes?*

Exercise 15. *Multiply your answers to the previous two problems to find the probability of both of the events described in the previous two problems occurring. This is a sort of p-value. What value of* p^* *would one need in order to accept the null hypothesis (that Dream was playing a fair, un-modified version of Minecraft)?*

The next few questions are meant to give you some intuition for how probable this event really is.

Exercise 16. *One of the luckiest moments ever recorded in a casino game was in 2009, when Patricia Demauro rolled a pair of dice 154 times consecutively at a craps table without ever rolling a 7. What is the probability of this event, and how does it compare to the probability that you found in Exercise 12?*

Exercise 17. *Suppose 10 billion humans are performing the following actions once per second for 10 years: killing 305 Blazes and bartering with 262 cleric villagers. What is the probability that none of them achieve what Dream did in only one try: getting at least 211 Blaze rods out of 305 and getting at least 42 pearl trades in 262 barters? Hint: count the number of seconds and multiply by the number of humans to find the number of trials. You'll likely need a scientific calculator like* `wolframalpha.com` *to find an answer with sufficient precision.*

Exercise 18. *Brawl Stars is a fun mobile game that I'm sure many of you play (certainly not during class though, right?). In Brawl Stars, there is a 2% chance of getting a Legendary Starr Drop and a 2.17% chance of getting a Legendary Brawler within a Legendary Starr Drop. In fact, the only way to get a Legendary Brawler from a Starr Drop is through a Legendary Starr Drop. Your friend has acquired n consecutive Legendary Brawlers from Starr Drops, where $n \geq 0$. What is the probability of acquiring at least n consecutive Legendary Brawlers from Starr Drops? How large does n need to be before this probability is less than the probability you derived in Problem 12?*

Exercise 19. *Finally, in your opinion, did Dream cheat?*

Challenge problems.

Exercise 20. *The chance of a runner to improve their own personal record in a race is q. What is the chance that they improve their record for the first time on their third race? How about on the nth race? Mathematicians call this the* ***geometric distribution****.*

Exercise 21. *Lucas starts flipping* $\mathrm{Ber}(q)$ *(defined earlier in the packet) until he sees r heads. What is the probability that Lucas sees exactly k tails before the rth head? Mathematicians call this the* ***negative binomial distribution****.*

Exercise 22. *On a sold-out flight, the first person to board the plane forgot which seat was his and chose a random seat. Subsequent passengers took their assigned seat if available, or a randomly chosen seat if not. When the last person boarded, there was only one seat left. What is the probability that this was actually the seat assigned to the last passenger?*

Exercise 23. *You are a contestant on a game show. There are three doors. Behind one door is a brand-new car. Behind the other two doors are goats. You are invited to choose one of the doors. Before opening the selected door, the show host opens one of the other two doors, revealing a goat. You can now either keep your original choice, or switch to the other unopened door. Which choice gives you a better chance of winning the car?*