THE AM-GM INEQUALITY

MATH CIRCLE (ADVANCED) 11/11/2012

The arithmetic mean (AM) and geometric mean (GM) of the numbers $a_1, a_2, \ldots, a_n \geq 0$ are given by

$$AM = \frac{a_1 + \cdots + a_n}{n} \text{ and } GM = \sqrt[n]{a_1 \cdots a_n}.$$

0) a) Calculate the AM and GM of the following sets of numbers:

i) $1, 2, 3, 6$

ii) $0, 4, 8, 20$

iii) $1, 3, 4, 7, 8$

iv) $4, 4, 4, 4$

b) What do you notice about AM compared to GM?

The AM-GM inequality states that if $a_1, a_2, \ldots, a_n \geq 0$, then

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \text{ with equality if and only if } a_1 = a_2 = \cdots = a_n.$$

2) a) Write out the AM-GM inequality for the numbers a, b.

b) Prove your statement in a).

3) Prove the following:

a) $\dfrac{x^2 + y^2}{2} \geq xy$ for any x, y.

b) $2(x^2 + y^2) \geq (x + y)^2$ for any x, y.

c) $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{x + y}$ for $x, y \geq 0$.

4) Prove that

a) $x^2 + y^2 + z^2 \geq xy + yz + zx$ for any x, y, z.

b) $(a + b)(a + c)(b + c) \geq 8abc$ for any a, b, c.

c) $x^2 + y^2 + 1 \geq xy + x + y$ if $x, y \geq 1$, (for a challenge prove it for any x, y).

d) $x^4 + y^4 + z^4 \geq xyz(x + y + z)$ for any x, y, z.

5) The sum of two non-negative numbers is 10. What is the maximum and minimum value of the sum of their squares?

6) Prove the AM-GM inequality for $n = 4$.

7) Prove that: (Hint: You may use the AM-GM inequality for any n.)

a) $a^4 + b^4 + 8 \geq 8ab$ for any a, b.

b) $(a+b+c+d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}) \geq 16$ for $a, b, c, d \geq 0$.

c) $3x^3 - 6x^2 + 4 \geq 0$ for $x \geq 0$.

d) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$ for $a, b, c \geq 0$.

8) a) Prove the AM-GM inequality for $n = 3$. Hint: Use 6)!

b)* Prove by induction that the AM-GM inequality holds for n a power of 2.

c)* Prove the AM-GM inequality for all n.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"