Intermediate 1 **March 3, 2024**

Mathematical Games

1 Nim

Problem 1.1. *You pick up 1 through 6 of 30 matches. The second player picks up 1 through 6 matches, and so on. The player who picks up the last match wins. How do you get to pick up the last match?*

Problem 1.2. *Problem 1.1, but this time the player who picks up the last match loses. How do you force the other player to pick up the last match?*

Problem 1.3. *To players pick up 1 through 4 of 27 matches until they are all picked up. You are the first player. To win, you must have an even number of matches at the end. How do you win this game?*

Problem 1.4. *There are two piles of pebbles. Players take turns picking up pebbles as follows:*

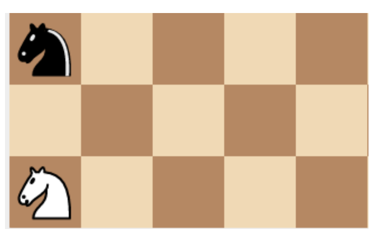
1. *You can pick up any or all the pebbles in one pile, or*
2. *You can pick up pebbles from both piles, but only if you take the same number from each pile.*

The player who takes the last pebble wins. Winning positions include $(1, 0)$ *(1 pebble in the first pile, none in the second) and* (n, n). *You pick up, respectively,* 1 *pebble (using rule 1) and* $2n$ *pebbles (using rule 2). In contrast,* $(1, 2)$ *is a losing position: no matter what you do, the other player can pick up all the remaining pebbles. Suppose the two piles have* m *and* n *pebbles respectively. Fill out the table below with whether the positions are winning (1) or losing (0).*

$m \setminus n$	0	1	2	3	4	5	6	7	8
0	0	1	1	1	1	1	1	1	1
1	1	1	0						
2	1	0	1						
3	1			1					
4	1				1				
5	1					1			
6	1						1		
7	1							1	
8	1								1

Problem 1.5. *Dan and Sam play a game on a convex polygon of 100001 sides. Each one draws a diagonal on the polygon in his turn. When someone draws a diagonal, it cannot have common points (except the vertices of the polygon) with other diagonals already drawn. The game finishes when someone can't draw a diagonal on the polygon following the rules; that person is the loser. If Dan begins, who will win, assuming both players play optimally?*

Problem 1.6. *Dan and Sam play a game on a* 5×3 *board. Dan places a White Knight on a corner and Sam places a Black Knight on the nearest corner. Each one moves his Knight in his turn to squares that have not been already visited by any of the Knights at any moment of the match. When someone can't move, he loses. If Dan begins, who will win, assuming both players play optimally?*



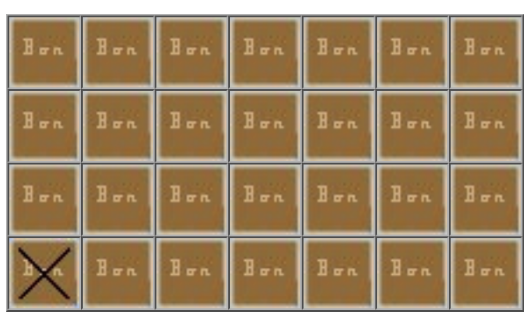
Problem 1.7. *Dan and Sam play a game on an* 8×8 *grid, on which each one chooses and puts, in his turn, a single piece. The pieces can be either* 1×2 *or* 2×1*. The pieces must not overlap and can't be partially outside of the grid. The first person who is unable to put a piece on the board in his turn following the rules loses. If Dan begins, who will win, assuming both players play optimally?*

Problem 1.8. *There is an even number of coins placed in a row. Two players take turns picking up coins, but they can only pick a coin at either end of the row. The coins can have values that are arbitrarily large or small. As the player going first, how can you guarantee to get at least as much money as the second player?*

Problem 1.9. *Dan and Sam take turns placing bishops on the squares of a chessboard, so that they cannot capture each other (the bishops can capture any other pieces of chess in either of the diagonal direction). The player who cannot move loses. Suppose Dan goes first. Who would wins if they play optimally?*

Problem 1.10. *There are n chips on the table. Two players, Player One (P1) and Player Two (P2), alternate removing chips from the table. On the first move, P1 can take any number of chips except for the whole pile. After that, the next player is not allowed to take more chips than the other player just did on the previous move. For example, if P1 made the first move removing two chips, P2 is allowed to remove either one or two chips. If P2 chooses to remove one chip, then on the next move, P1 is only allowed to remove one chip. The player to take the last chip wins. Choosing to play either P1 or P2 is a part of the strategy. Find a way to always win. (Hint: it helps to use the binary notation for the number of chips analyzing the game.)*

Problem 1.11. *The game of Chomp is like Russian Roulette for chocolate lovers. A move consists of chomping a square out of the chocolate bar along with all squares to the right and above. Players alternate moves. The lower left square is poisoned though and the player forced to chomp it loses. How do you win as the first player?*



2 Impartial Game

All of the games we have considered above are called impartial (combinatorial) games.

Definition 1. *An impartial (combinatorial) game is a game in which*

- *Two players alternate turns;*
- *There is no secret information (i.e. no face down cards);*
- *All actions are deterministic (i.e. no throwing dice);*
- *It's guaranteed to finish in finite number of turns;*
- *Both players follow the exact same set of rules;*
- *In "normal play", the first player who cannot move loses, and the other player wins. In "misère play", the first player who cannot move wins, and the other player loses.*

Problem 2.1. *Which of the following games are impartial games?*

- *Chess*
- *Checkers*
- *Poker*
- *Basketball*

Problem 2.2. *Design an impartial game that is not mentioned in this handout. Is there anyone who is guaranteed to win if they play optimally?*

Definition 2. *Suppose G_1, G_2 are two impartial games. We define $G_1 + G_2$ to be a game in which each player take turns to make a move in either G_1 or G_2, and the first player who cannot move in both of the games loses.*

For example, if G_1 is the game in Problem 1.6, and G_2 is the game in Problem 1.9, then $G_1 + G_2$ is a game in which a 5×3 board and a 8×8 chessboard are placed side by side. In the first board, they play the game described in Problem 1.6, and on the second board they play the game in Problem 1.9. At each turn, Dan and Sam can only choose one board to move, and whoever cannot move on both boards loses.

Problem 2.3. *In this example, who can win $G_1 + G_2$ if they both play optimally?*

Problem 2.4. *Justify that the sum of two impartial games is still an impartial game.*

Problem 2.5. *Assuming all players play optimally. If the first player can win G_1, who can win $G_1 + G_1$?*

Problem 2.6. *Let G_1 be the game in Problem 1.1 and G_2 be the game in Problem 1.6. Suppose you are the second player. How do you win $G_1 + G_2$?*