Number Theory

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1 Warm Up

Two integers are inserted into the list $3, 3, 8, 11, 28$ to double its range. The mode and median remain unchanged. What is the maximum possible sum of two additional numbers?

(A) 56 (B) 57 (C) 58 (D) 60 (E) 61

How many positive integers can fill the blank in the sentence below?
"One positive integer is ______ more than twice another, and the sum of the two numbers is 28."

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

A number is called flippy if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by 15?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 8

2 Review

Prime Numbers are numbers that **don't have** any other factors other than one and itself. Some examples are 2, 3, and 89.
Composite Numbers are numbers that **have** other factors other than one and itself. Examples include 4, 15, and 120.

2.1 Exercise 1

Using your knowledge of divisibility from the first week, find all the prime numbers in the following grid. Use the following instructions:

1. Cross out 1 by **Shading** in the box completely. One is neither prime nor composite.
2. Use a forward **Slash** to cross out all multiples of 2 (starting with 4 since 2 is prime).
3. Use a backward **Slash /** to cross out all multiples of 3 (starting with 6 since 3 is prime).
4. Multiples of 4 have been crossed out already (multiples of 2).
5. Draw a **Square** on all multiples of 5 (starting with 10 since 5 is prime).
6. Multiples of 6 have been crossed out already (multiples of 2 and 3).
7. **Circle** all multiples of 7 (starting with 14 since 7 is prime).
8. Multiples of 8 have been crossed out already (multiples of 2).
9. Multiples of 9 have been crossed out already (multiples of 3).
10. Multiples of 10 have been crossed out already (multiples of 2 and 5).

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

What numbers did you have left? Write them here.

You just found all the prime numbers less than 100! The method we used was called the *Sieve of Eratosthenes*. It's an ancient algorithm that is cool to know and use, but don't worry: you won't need to remember the name of it.

3 Factors

A *factor* of an integer n can be defined as an integer that can be multiplied by another integer to get this number n.

3.1 Divisibility Rules

A number is divisible by:
2: If it is even.
3: If the sum of its digits is divisible by 3.
4: If the number formed by the last two digits is divisible by 4.
5: If the one's digit is 5 or 0.
6: If it is divisible by 2 and 3 (all even multiples of 3).
7: There is no *good* trick for 7, just do the division.
8: If the number formed by the last three digits is divisible by 8.
9 If the sum of the digits is divisible by 9.
10: If the last digit is a 0.
11: If the difference of the sum of alternative digits of a number is divisible by 11

Circle the numbers that are divisible by 11. Do not use technology.

2002 222 704 320 342 9281701

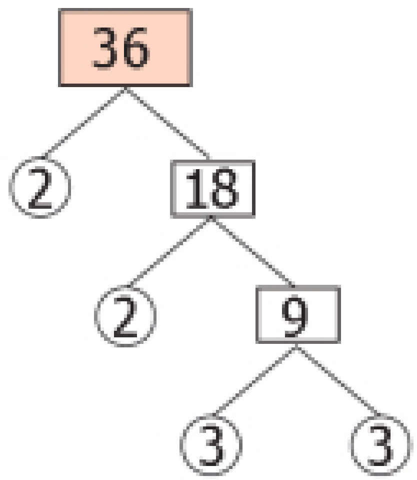
3.1.1 Divisibility Exercises

1. Determine whether the following numbers are divisible by 6 **using divisibility rules**.
   (a) 264
   (b) 975
   (c) 12,560
2. Determine whether the following numbers are divisible by 72 (*hint*: 72 is $8 * 9$) **using divisibility rules**.
   (a) 924
   (b) 1,284
   (c) 14,790

3.2 The Fundamental Theorem of Arithmetic

The Fundamental Theorem of Arithmetic states that every positive integer has a unique prime factorization. For example, $5544 = (2)^3(3)^2(7)(11)$. There is no other way to factor 5544 into a product of primes.

The easiest way to find the prime factorization of a number is to create a factor tree. Here is an example:



Try it yourself! Use a factor tree to find all the factors of **a)** 84 and **b)** 520.

3.3 What can you do with factors?

There will be many different questions the AMC test might ask where factors will be needed, but here are some of the most common usages.

3.3.1 Number of Factors

The sum of the factors of an integer n, where a, b, c etc. represent the prime factors of n and $(a)^p(b)^q(c)^r$... is the prime factorization of n, is

$$\frac{a^{p+1}-1}{a-1} \cdot \frac{b^{q+1}-1}{b-1} \cdot \frac{c^{r+1}-1}{c-1} \ldots$$

Try finding the sum of the factors of the following numbers: **a)** 375 **b)** 360.

3.3.2 Product of Factors

The product of the factors of an integer n, where S is the sum of its factors as defined above, is

$$n^{\frac{S}{2}}$$

Try finding the product of the factors of the following numbers: **a)** 375 **b)** 360.

4 Practice

Lets practice doing word problems and number theory with the following past AMC test problems!

Let n be the largest integer that is the product of exactly 3 distinct prime numbers d, e, and $10d + e$, where d and e are single digits. What is the sum of the digits of n?

(A) 12 (B) 15 (C) 18 (D) 21 (E) 24

What is the probability that an integer in the set $\{1, 2, 3, ..., 100\}$ is divisible by 2 and not divisible by 3?

(A) $\frac{1}{6}$ (B) $\frac{33}{100}$ (C) $\frac{17}{50}$ (D) $\frac{1}{2}$ (E) $\frac{18}{25}$

What is the units digit of 13^{2003}?

(A) 1 (B) 3 (C) 7 (D) 8 (E) 9

A base-10 three digit number n is selected at random. Which of the following is closest to the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals?

(A) 0.3 (B) 0.4 (C) 0.5 (D) 0.6 (E) 0.7

Let n be a 5-digit number, and let q and r be the quotient and the remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

(A) 8180 (B) 8181 (C) 8182 (D) 9000 (E) 9090

How many positive integers n satisfy the following condition: $(130n)^{50} > n^{100} > 2^{200}$?

(A) 0 (B) 7 (C) 12 (D) 65 (E) 125

How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

(A) 41 (B) 42 (C) 43 (D) 44 (E) 45

How many positive cubes divide $3! \cdot 5! \cdot 7!$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property?

(A) 5 (B) 7 (C) 9 (D) 10 (E) 19

For how many positive integers n does $1 + 2 + ... + n$ evenly divide from $6n$?

(A) 3 (B) 5 (C) 7 (D) 9 (E) 11

Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T?

(A) 166 (B) 333 (C) 500 (D) 668 (E) 1001

For each positive integer $n > 1$, let $P(n)$ denote the greatest prime factor of n. For how many positive integers n is it true that both $P(n) = \sqrt{n}$ and $P(n + 48) = \sqrt{n + 48}$?

(A) 0 (B) 1 (C) 3 (D) 4 (E) 5