Exploration Circle: Astronomy, Cartography, and Navigation

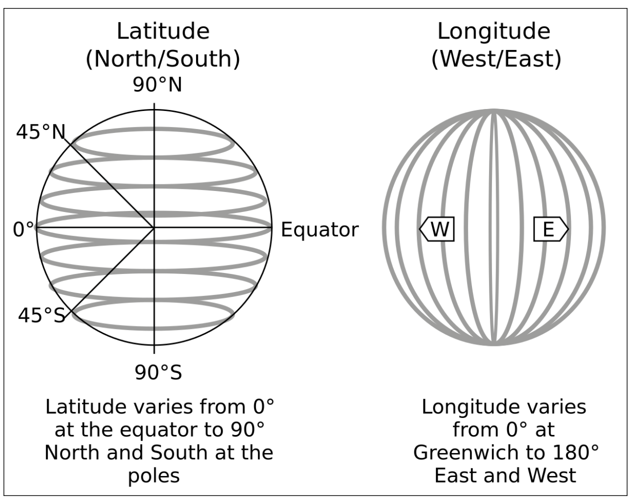
Before starting this handout, please obtain or craft a ruler. You can cut out the ones on the back. I encourage you to work in groups. Perform the following instructions:

- Go outside, somewhere flat.
- Get a stick (or ideally a plumb line[1]).
- Measure the vertical height of the stick above the ground.
- Measure the length of its shadow.
- Record the exact time of measurement.

You will need to perform this measurement again in about an hour. If you do not get to Problem 5 in that time, please remember to come back to this and do your second measurement before the sun sets.

1 Latitude & Longitude

Locations on earth are typically denoted using coordinates of latitude and longitude. Given a point on the surface of the earth, latitude[2] measures the angle from the centre of the earth with the plane of the *equator*, and longitude measures the angle from the *prime meridian* which everyone agreed would run through Greenwich, England for some reason[3].



For the purposes of this packet, we will use the same notation and units as Google Maps:

$$(x,\ y)$$

- $-90^\circ \leq x \leq 90^\circ$ is latitude in decimal degrees, positive is north of equator.
- $-180^\circ \leq y \leq 180^\circ$ is longitude in decimal degrees, positive is east of prime meridian

[1] A line with a heavy thing at the end of it used as a reference for verticality with respect to gravity. Traditionally, the heavy thing on the end was a made of lead so that's why it's called a plumb line/bob.

[2] lines of longitude are also called *parallels* for obvious reasons

[3] That's why all time zones are measured relative to Greenwich Mean Time (GMT) or, more formally, UTC+00:00.

For example, the coordinates of the Gastown Steam Clock in Vancouver, Canada are 49.28460449890565, -123.1088179456726. The first number is positive because Vancouver is in the Northern Hemisphere. The second number is negative because Vancouver is in the Western Hemisphere.

World Robinson Projection Map with Country Outlines

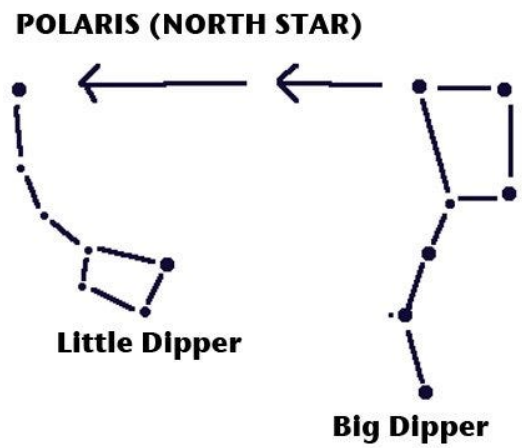
1. Using only the map above, guess what landmark is at 27.98847718790856, 86.92482948890476. Enter your coordinates into your phone to confirm your guess.

   Hint: The lines of longitude and latitude aren't labelled on this map, try to figure out what the labels should be.

2 Latitude: Sextants

Before GPS was invented, people used celestial bodies to determine their coordinates. One tool commonly use to do this is called a *sextant*. A sextant is really just a fancy protractor that you can use to measure the angle between faraway objects. You can make your own using household supplies or even just your phone.

Conveniently, the north star *Polaris* is located almost exactly on the earth's axis of rotation[4]. Polaris is also very easy to spot if you can find the Big Dipper. So, for little explorers in the Northern Hemisphere (like you), determining latitude on a clear night is easy!



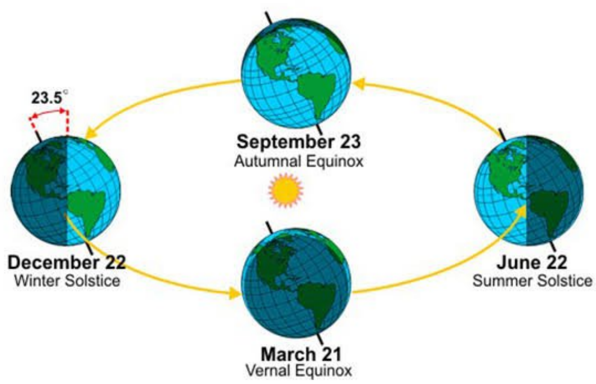
2. What is the relationship between latitude and the angle of Polaris above the horizon? (*Hint: Draw a diagram. What simplifying assumption should you make?*) Build a sextant, go outside, find Polaris, and determine your latitude.

   Note: In the likely event the north star is not currently visible, we'll just give you the answer. I encourage you to try this at night though.

[4]Have you ever seen those composite photographs where the stars trace out concentric circles in the sky? The star in the middle that doesn't seem to move is Polaris.

What if it's daytime? What if you find yourself in the Southern Hemisphere? Fear not, we can also use the sun! The angle the sun makes with the horizon at solar noon depends on the date and latitude of observation. So, if we know the date and solar altitude angle, we can determine our latitude.

3. *Noon* and *solar noon* aren't quite the same thing. *Noon* is just when your watch says that it's 12 o'clock during the daytime. *Solar noon* is when the sun is at it's highest for a given day. Using a magnetic compass and some sticks, how can you determine solar noon?



4. Calculate your latitude again but using the sun. Of course, your answer should be the same as in Problem 2. You will need to know the following:

   - I've already measured the solar altitude angle at solar noon for you. Ask your instructors for today's measurement.
   - The earth's axis of rotation is tilted by 23.5°, see diagram above.
     - This is why the days are long and warm in the summer but short and cold in the winter. The length of days barely changes near the equator, but people living near the poles will have months of continuous daylight in the summer and months of continuous darkness in the winter.
   - Winter and Summer solstice are respectively the shortest and longest days of the year. Hopefully, you can see from the diagram why that is. The last winter solstice for the Northern Hemisphere was December 22, 2023.
   - 2024 is a leap year.
   - Trigonometry.

   Hint: Draw a diagram. Assume that the sun is really, really far away, because it is. It's easier to first ignore the earth's axial tilt and then figure out how to correct for it afterwards.

Do your work for Problem 4 on this page.

5. The *subsolar point* is the point on the surface of the earth at which a vertical pole casts no shadow since the sun is directly overhead. You can use the following website to check the subsolar point at any date and time: https://www.timeanddate.com/worldclock/sunearth.html

   In this question, you will determine your location on a map (next page) by calculating your distance from the subsolar point at two points in time.

   (a) Remember the measurements you took before you started this packet? Now go outside and do it again so that you have two data points.

   (b) From your measurements, determine the solar altitude angles for each measurement.

   (c) From the solar altitude angles, determine how far away you were from the subsolar point for each measurement. Use nautical miles (1 NM = 1.151 mi) as your units. The circumference of the earth is approximately $60 \times 360 = 21600$ NM. Thus, 1 degree of latitude/longitude corresponds to a change in 60 NM.

   (d) Using the linked website, enter the correct date and times and mark the two subsolar points on the map. Ask an instructor or neighbour to borrow a computer/phone if needed.

   (e) Can you figure out the rest on your own? Using the information you have so far, how can you determine your location on the map? Use your ruler. Read the footnote[5]. A compass helps draw circles but not strictly necessary.

   (f) What are some possible ambiguities and how can they be mitigated?

[5] Since I couldn't find an appropriate map that also included a scale, you'll have to create your own. Look at Saskatchewan, the Canadian rectangle. Its southern border runs along the 49th parallel and its northern border is on the 60th parallel. Combined with what I told you about nautical miles, you can approximate the scale of this map. Admittedly, this is a Mercator projection which has certain drawbacks that you will explore in the next section. Your measurements were probably terrible though, so it doesn't matter ☺

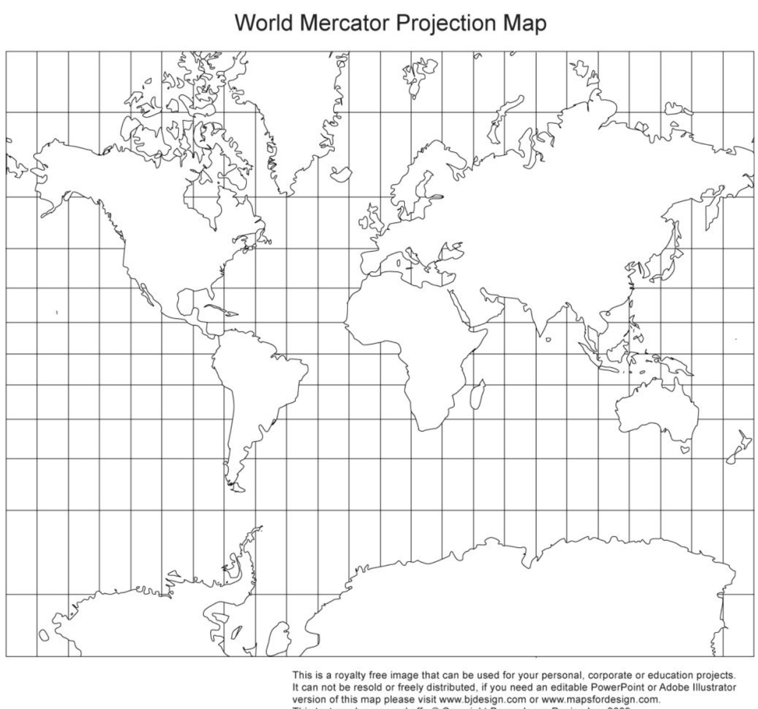
3 Longitude: Chronometers

Determining longitude by celestial navigation typically requires highly accurate time-keeping[6]. In the early days of naval exploration, variable weather and turbulent seas rendered the pendulum clocks of the time useless. Due to frequent naval disasters caused by an inability to accurately determine longitude, the 1714 British Parliament famously offered up to £20,000 (£3.35 million in 2024) for anybody who could solve the issue. The prize was most notably claimed by carpenter and clock-maker John Harrison who built a mechanical clock resistant to maritime conditions using a balance-wheel instead of a pendulum.

6. Imagine yourself a 19th century navigator equipped with a chronometer which has been calibrated perfectly to the time in Greenwich. At solar noon, you observe that your chronometer reads 19:52:00. What is your longitude?

4 The Mercator Projection

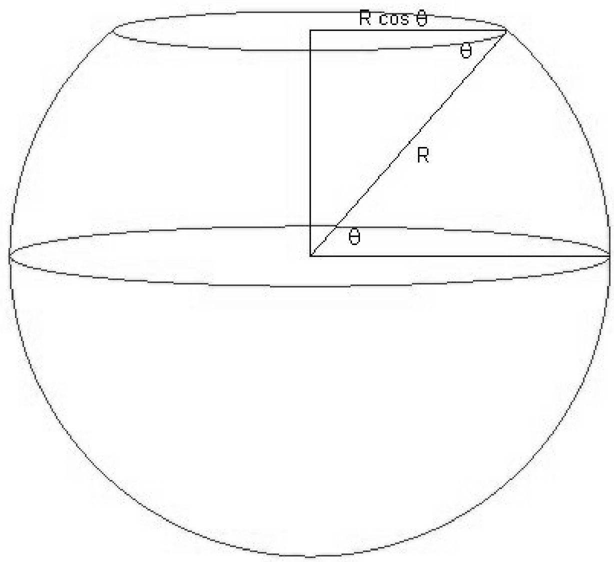
The **Mercator Projection** was invented by Gerardus Mercator, a mapmaker in the 1500s. As previously mentioned, the Mercator projection depicts the globe so sailors can look at the map, draw a straight line between two points, and travel at the angle formed by this line. It is one of the most used projections in today's maps. A common misconception is that a Mercator projection is similar to if a globe was placed in a cylinder, a light was projected from its center onto the cylinder, and the result was unraveled. A more accurate intuition is to imagine that the globe is a spherical balloon that expands inside of a cylinder, sticking to it wherever it touches.



[6] Even before the invention of maritime chronometers, alternative methods of determining longitude often involved determining the time in Greenwich by tracking the motion of the moon against distant stars and comparing with an almanac. Calculating the tables in these almanacs was extremely difficult in its own right.

7. Every map projection is designed to accomplish certain goals while making certain sacrifices. What are some pros and cons of the Mercator projection?

Edward Wright was the mathematician who explained the Mercator projection. Assume that the Earth is a sphere, θ is the latitude angle.



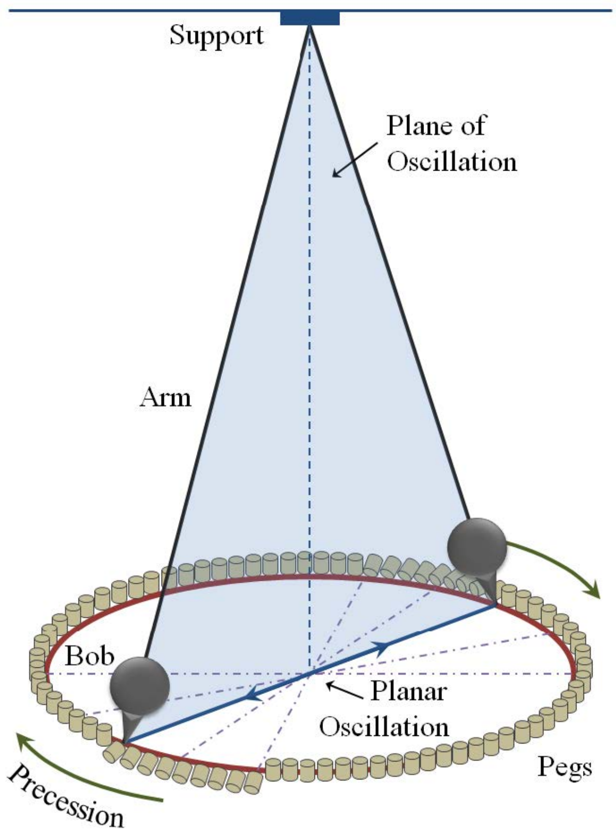
8. Parallels decrease in circumference as you get near the poles, but a Mercator projection depicts them with all the same length. By what factor is the length of a parallel at latitude θ stretched? We say that the equator is stretched by a factor of 1 since the Mercator projection does not distort the equator.

9. *(Bonus):* Let $F(\theta)$ be the distance on the Mercator map between the parallel at latitude θ and the equator. Find $F(\theta)$. *Hint: Try to find the relationship between the angle and the distance when we vary the angle by a very small quantity.*

5 Foucault Pendulum

To demonstrate the rotation of the Earth, French physicist Léon Foucault devised the *Foucault Pendulum*. It is a pendulum mounted on a frictionless bearing that effectively isolates the pendulum's swinging from the rotation of the earth. Perhaps you've seen the one at Griffith Observatory.

As the earth rotates, the pendulum keeps swinging in the same direction, but everything around it rotates with the earth. From the perspective of us, standing on the earth, it looks like the pendulum is slowly rotating. We can put pegs around the pendulum to track its rotation.



10. At the the poles (latitude = $\pm 90°$), the pendulum makes one full rotation per day (360°/day). At the equator (latitude = 0°), it doesn't rotate at all (0°/day). For everything in between, the angular speed changes with latitude according to a very simple relationship. Determine this relationship.

11. Using your answers to Problems 2 and 9, how long does it take the Foucault pendulum at Griffith Observatory to complete a full rotation? Check your answer by going to Griffith Observatory.

12. Assume that a year is exactly 365 days long. The Earth spins about its axis in the same direction it rotates around the Sun. How many days would there be in a year if the Earth spun around its axis in the opposite direction?

6 Spherical Geometry

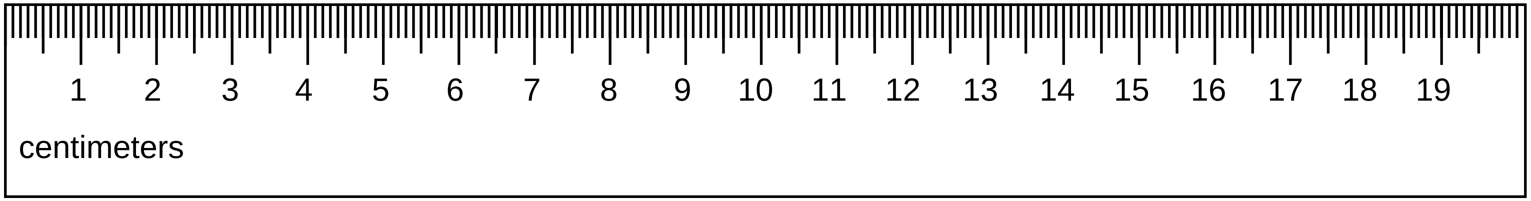
13. A tetrahedron has angles α, β, and γ. Prove that $\alpha + \beta > \gamma$.

14. Use Problem 14 to show that the shortest path between two different points on a sphere is always an arc of a *great circle*. A great circle of a sphere has the largest radius of any circle that can be drawn on that sphere; in essence, a great circle is a circumference of a sphere.

15. A *biangle*[7] on a unit sphere has the angle size α, measured in radians. Find the area of the biangle.

16. A triangle on a unit sphere has the angle sizes α, β, and γ, measured in radians. Find the area of the triangle.

[7]yes, a two-sided polygon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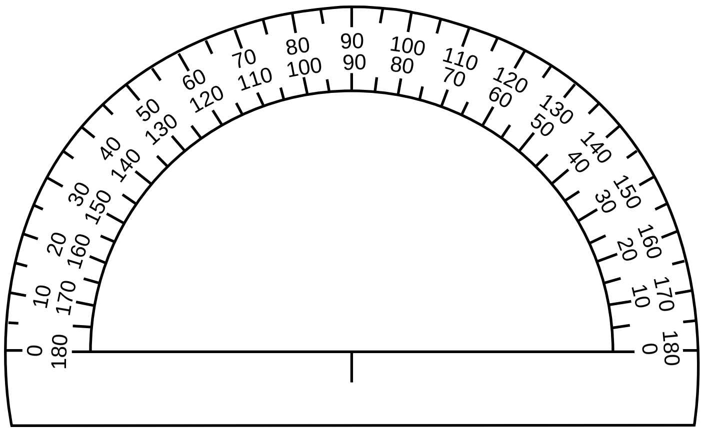
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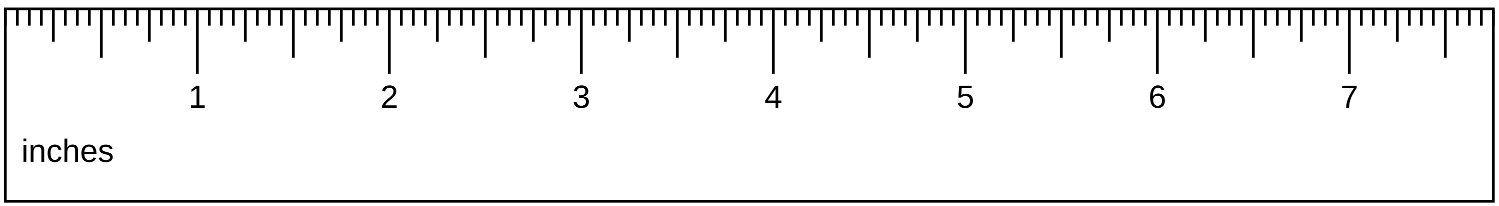
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