ORMC AMC 10/12 Group
Week 9: Algebra

March 3, 2024

1 Warm-up

1. For real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, expand the following products:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} \qquad\qquad + \sum_{1\le k<j\le n}$$

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \sum_{k=1}^{n} \qquad\qquad + \sum_{1\le k<j\le n}$$

2. Show that for any $1 \le k < j \le n$,

$$2a_k a_j b_k b_j \le a_k^2 b_j^2 + a_j^2 b_k^2$$

3. Conclude that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

   This is the **Cauchy-Schwarz Inequality**.

4. Using the base form of AM-GM, $\frac{a+b}{2} \ge \sqrt{ab}$ for positive real numbers a, b, show inductively that when n is a power of 2,

$$\frac{a_1 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 \cdots a_n}$$

5. Conclude that AM-GM holds for all n by showing that if it holds for n, then it holds for $n-1$.

2 Review of Theorems/Techniques

There are many different types of AMC problems which are broadly categorized as "Algebra", but generally they fall into one or more of a few subcategories:

1. **Solving Equations/Inequalities**
    - For the most part, these questions are pretty basic and can be solved, at a high level, by using the basic (system-of-) equation solving techniques like substitution and elimination
    - The AMC adds complexity to these questions by requiring additional techniques to solve the equations, from various topics we have covered, like counting, modular arithmetic, and trigonometry.

2. **Factoring**
    - Usually, when we say "factoring", we're referring to factoring or multiplying out polynomials. However, as you'll see in the worksheet, the following identities show up in all sorts of situations:
        - **Difference of Squares:** $\boxed{a^2 - b^2 = (a-b)(a+b)}$
        - **Difference of Powers:** $\boxed{a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}}$
          (Works for any integer n)
        - **Sum of Powers:** $\boxed{a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 + \cdots + (-1)^n b^{n-1}}$
          (Works only for *odd* integers n) Note that this can be derived from difference of powers, by factorizing $a^n - (-b)^n$ for n odd.
        - **Binomial Theorem:** $\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}$

          Especially important to remember the factorizations for $n = 2, 3, 4$:

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

$$\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

$$\boxed{(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4}$$

        - **Simon's Favorite Factoring Trick:** $\boxed{ab + a + b + 1 = (a+1)(b+1)}$

3. **Functions:** another very broad category, but these types of questions generally involve:
    (a) **Recursion**: given a recurrence relation, add various iterations of itself together, to solve for a function explicitly
    (b) **Solving Functions**: given restrictions on a function, find the function (or some other properties of it)
    (c) **Graphing**: Find shapes/areas/intersections of various graphs; often helps to use translations, rescaling, and other basic transformations to your advantage.
    (d) **Logarithms**: Remember the properties of logarithms:

$$\textbf{Definition of Log}: a = \log_b(b^a) = b^{\log_b(a)}$$
$$\textbf{Product Rule}: \log(a \cdot b) = \log(a) + \log(b)$$
$$\textbf{Division Rule}: \log(a/b) = \log(a) - \log(b)$$
$$\textbf{Exponent Rule}: \log(a^b) = b\log(a)$$
$$\textbf{Change of Base}: \log_b(a) = \log(a)/\log(b)$$

3 Exercises

1. **(2007 AMC 12B #5)** The 2007 AMC 12 contests will be scored by awarding 6 points for each correct response, 0 points for each incorrect response, and 1.5 points for each problem left unanswered. After looking over the 25 problems, Sarah has decided to attempt the first 22 and leave the last 3 unanswered. How many of the first 22 problems must she solve correctly in order to score at least 100 points?

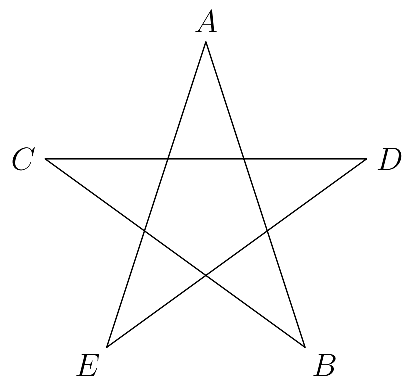
2. **(2006 AMC 12A #12)** A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?

3. **(2006 AMC 12A #11)** Which of the following describes the graph of the equation $(x+y)^2 = x^2+y^2$?
   (A) the empty set (B) one point (C) two lines (D) a circle (E) the entire plane

4. **(2005 AMC 12A #10)** A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

5. **(2005 AMC 12A #19)** A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. If the odometer now reads 002005, how many miles has the car actually traveled?

6. **(2005 AMC 12A #13)** In the five-sided star shown, the letters A, B, C, D and E are replaced by the numbers 3, 5, 6, 7 and 9, although not necessarily in that order. The sums of the numbers at the ends of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$ form an arithmetic sequence, although not necessarily in that order. What is the middle term of the arithmetic sequence?



7. **(2020 AMC 10A #16)** A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0),(2020,0),(2020,2020)$, and $(0,2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x,y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

8. **(2015 AMC 10A #15)** Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

9. **(2008 AMC 12A #19)** In the expansion of

$$\left(1+x+x^2+\cdots+x^{27}\right)\left(1+x+x^2+\cdots+x^{14}\right)^2,$$

what is the coefficient of x^{28}?

10. **(2014 AMC 10A #15)** David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

11. **(2013 AMC 12A #16)** A, B, C are three piles of rocks. The mean weight of the rocks in A is 40 pounds, the mean weight of the rocks in B is 50 pounds, the mean weight of the rocks in the combined piles A and B is 43 pounds, and the mean weight of the rocks in the combined piles A and C is 44 pounds. What is the greatest possible integer value for the mean in pounds of the rocks in the combined piles B and C?

12. **(2005 AMC 12A #20)** For each x in $[0, 1]$, define

$$\begin{aligned} f(x) &= 2x, && \text{if } 0 \le x \le \tfrac{1}{2}; \\ f(x) &= 2 - 2x, && \text{if } \tfrac{1}{2} < x \le 1. \end{aligned}$$

Let $f^{[2]}(x) = f(f(x))$, and $f^{[n+1]}(x) = f^{[n]}(f(x))$ for each integer $n \ge 2$. For how many values of x in $[0, 1]$ is $f^{[2005]}(x) = \frac{1}{2}$?

13. **(2005 AMC 12A #21)** How many ordered triples of integers (a, b, c), with $a \ge 2$, $b \ge 1$, and $c \ge 0$, satisfy both $\log_a b = c^{2005}$ and $a + b + c = 2005$?

14. **(2018 AMC 10A #21)** Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy-plane intersect at exactly 3 points?

(A) $a = \frac{1}{4}$ **(B)** $\frac{1}{4} < a < \frac{1}{2}$ **(C)** $a > \frac{1}{4}$ **(D)** $a = \frac{1}{2}$ **(E)** $a > \frac{1}{2}$

15. **(2018 AMC 10B #20)** A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

16. **(2023 AMC 10B #14)** How many ordered pairs of integers (m, n) satisfy the equation $m^2+mn+n^2 = m^2n^2$?

17. **(2015 AMC 10A #20)** A rectangle with positive integer side lengths in cm has area A cm^2 and perimeter P cm. Which of the following numbers cannot equal $A + P$?

    (A) 100 **(B)** 102 **(C)** 104 **(D)** 106 **(E)** 108

18. **(2012 AMC 10A #19)** Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

19. **(2012 AMC 10A #24)** Let a, b, and c be positive integers with $a \geq b \geq c$ such that $a^2 - b^2 - c^2 + ab = 2011$ and $a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997$.

    What is a?

20. **(2013 AMC 12B #17)** Let a, b, and c be real numbers such that

$$a + b + c = 2, \text{ and}$$
$$a^2 + b^2 + c^2 = 12$$

What is the difference between the maximum and minimum possible values of c?

21. **(2020 AMC 10A #21)** There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}.$$

What is k?

22. **(2006 AMC 12A #24)** The expression

$$(x + y + z)^{2006} + (x - y - z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

23. **(2006 AMC 12A #21)** Let

$$S_1 = \{(x, y) | \log_{10}(1 + x^2 + y^2) \leq 1 + \log_{10}(x + y)\}$$

and

$$S_2 = \{(x, y) | \log_{10}(2 + x^2 + y^2) \leq 2 + \log_{10}(x + y)\}$$

What is the ratio of the area of S_2 to the area of S_1?

24. **(2006 AMC 12A #18)** The function f has the property that for each real number x in its domain, $1/x$ is also in its domain and

$$f(x) + f\left(\frac{1}{x}\right) = x$$

What is the largest set of real numbers that can be in the domain of f?

(A) $\{x|x \neq 0\}$ (B) $\{x|x < 0\}$ (C) $\{x|x > 0\}$ (D) $\{x|x \neq -1 \text{ and x} \neq 0 \text{ and x} \neq 1\}$ (E) $\{-1, 1\}$

25. **(2008 AMC 12B #23)** The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?

26. **(2009 AMC 12A #24)** The tower function of twos is defined recursively as follows: $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Let $A = (T(2009))^{T(2009)}$ and $B = (T(2009))^A$. What is the largest integer k for which

$$\underbrace{\log_2 \log_2 \log_2 \ldots \log_2 B}_{k \text{ times}}$$

is defined?

27. **(2009 AMC 12A #25)** The first two terms of a sequence are $a_1 = 1$ and $a_2 = \frac{1}{\sqrt{3}}$. For $n \geq 1$,

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}.$$

What is $|a_{2009}|$?