Intermediate 1 **Feb 25, 2024**

Game Theory

1 Warm-Up

Problem 1.1. *Two players take turns putting round chips of the same size on a round table. Originally, the table is empty. The chips are not allowed to overlap or to stick outside of the table. The first player who is unable to make a move loses. If both players play optimally, who will win this game? Find a winning strategy.*

Problem 1.2. *A crew of pirates has the following hierarchy:*

$$A > B > C > D > E > F > G > H > I > J$$

A is the captain, B is the second in the hierarchy, and so forth. They have to split 1,000 gold coins, all of the same value, between the crew members. The way they share booty works as follows: the captain proposes a division scheme and then the pirates vote. It the majority votes for the scheme or if there is a draw, the scheme goes ahead. If the majority votes against the scheme, they kill the captain, the next in the hierarchy becomes the new captain, and the algorithm goes for the next run. All *the pirates are infinitely smart and infinitely ruthless. They totally distrust one another, so they cannot form alliances, trust promises, etc. "Infinitely ruthless" means the following: if a pirate has to choose between the situation that (1) he gets some number of coins and the current captain stays alive; and (2) he gets the same number of coins as in (1), but the captain is killed; the pirate will choose (2). What is the maximal number of coins the captain can keep to herself and stay alive?*

Problem 1.3. *Consider the same same situation, but this time the pirates are kind-hearted by nature; they are pirates by necessity, not choice. Being infinitely rational, they totally distrust one another, so they cannot form alliances, trust promises, etc. "Kind-hearted" means the following: if a pirate has to choose between the situation that (1) he gets some number of coins and the current captain stays alive; and (2) he gets the same number of coins as in (1), but the captain is killed; the pirate would choose (1).*

2 What is Game Theory?

Welcome to an exploration of Game Theory, a fascinating field that lies at the intersection of mathematics, economics, and psychology. Game Theory is the study of strategic decision-making, where the outcome of your choices depends not only on what you decide but also on the decisions of others. It's not limited to traditional"games" but applies to a wide range of competitive and cooperative situations—from sports and politics to negotiations and everyday choices. First, let's define what we really mean by a "game".

We say a **game** has the following properties:

1. There are at least two players (individuals, companies, nations, animals, . . .).
2. Each player has a number of possible strategies, which they may choose to follow.
3. The strategies chosen by each player determine the outcome of the game.
4. Associated to each possible outcome of the game is a collection of numerical payoffs, one to each player (each payoff represents the value of the outcome to the player).

A couple of things to keep in mind:

- Game Theory is the study of how players should rationally play games. This means that each player tries to maximize their own payoff, regardless of what the other person's payoff is.

- Each player has some control over the outcomes, since their choices will influence it.
- Games are not restricted to the typical ones, such as chess or poker (although both are great examples) they extend to:
  - Companies pursuing corporate strategies.
  - Political candidates trying to win an election.
  - Nations maneuvering in the international arena.
  - Auctions (of which there are several types).
  - Matrix games (pure, mixed).

For now, we only consider **pure games**, or games where players must choose an action deterministically. We will explore **mixed games**, or games where the player is allowed to randomize their action, in a future section.

Example 1 (Prisoner's Dilemma)**.** *The prisoner's dilemma is a classic game theory problem. The situation is as follows: you and your partner were arrested for a crime and are now being questioned, separately, about a more serious crime that you are suspected to have committed. Each of you can either stay silent or confess to the larger crime. If both of you stay silent, each of you gets a year in prison for the smaller crime. If both of you confess, each of you gets 5 years in prison for the serious crime. If, however, you confess and your partner stays silent, you are set free immediately and your partner gets 10 years in prison. Similarly, if you stay silent but your partner confesses, you are sentenced to 10 years in prison while your partner is set free immediately. Would you confess or stay silent? Would your action depend on what your partner does?*

The payoffs of a two-player game can be represented by a bimatrix, where each element of the matrix is a 2-tuple representing the payoff to each player. This is known as the **payoff matrix** of the game. Let's see what the payoff matrix of the prisoner's dilemma looks like:

Actions	Silent	Confess
Silent	(-1, -1)	(-10, 0)
Confess	(0, -10)	(-5, -5)

We read the table as follows: Player 1 (you) chooses a row action, and Player 2 (your partner) chooses a column action. Then, given the row and column in the payoff matrix, Player 1's payoff is the first element of the tuple, and Player 2's payoff is the second. Now, let's figure out our best strategy to the prisoner's dilemma:

1. If our partner stays silent:

   (a) If we stay silent too, the payoff matrix entry is (-1, -1), so our payoff is -1.

   (b) If we confess, the payoff matrix entry is (0, -10), so our payoff is 0.

   So, if we know our partner is going to stay silent, we should confess and get out scot-free! (Remember that we are selfish criminals and don't care about our partner's suffering. After all, there is no honor among thieves!)

2. If our partner confesses:

(a) If we stay silent, the payoff matrix entry is (-10, 0), so our payoff is -10.

(b) If we confess too, the payoff matrix entry is (-5, -5), so our payoff is -5.

So, if we know our partner is going to confess, we might as well confess too and minimize our inevitable prison sentence!

Notice that regardless of our partner's action, our best strategy is to always confess! This is called a **dominant strategy**, as it is our best course of action regardless of the other player's action. Since confessing is also the dominant strategy for our partner, we will both end up confessing and getting 5 years in prison, even though we could have both stayed silent and gotten away with a 1 year sentence.

Problem 2.1. *United Airlines (Player 1) and American Airlines (Player 2) are deciding what fare to set for a one-way ticket from Los Angeles to New York. Based on some market research, they both know the following payoff matrix for their profit (in millions of dollars). Figure out the dominant strategy for each of the airlines, and what their final profits will end up being.*

Fares	*\$500*	*\$200*
\$500	*(50, 100)*	*(-100, 200)*
\$200	*(150, -200)*	*(-10, -10)*

3 Nash Equilibria

Notice that, in both of the previous games, both players could have ended up with better payoffs if they could mutually agree to play different strategies. However, since they are unable to collude, they choose the strategy that suits them best regardless of the other player's action. A **Nash equilibrium** (named after mathematician John Nash), is a situation where, given all the current strategies of the other players, a player has no incentive to deviate from their current strategy.

Problem 3.1. *Find the (pure) Nash equilibrium to the following game:*

Strategy	*L*	*C*	*R*
T	*(1, 0)*	*(1, 3)*	*(3, 0)*
M	*(0, 2)*	*(0, 1)*	*(3, 0)*
B	*(0, 2)*	*(2, 4)*	*(5, 3)*

Problem 3.2. *Is it possible for a game to have more than one (pure) Nash equilibrium? If so, construct such a game. If not, justify why not.*

Problem 3.3. *Is it possible for a game to have no (pure) Nash equilibrium? If so, construct such a game. If not, justify why not.*

A **zero-sum game** is a matrix game where the payoffs for each possible outcome sum to 0.

Problem 3.4. *Consider the following payoff matrix. Verify that the game is indeed zero-sum.*

Strategy	*D*	*E*	*F*
A	*(-1, 1)*	*(0, 0)*	*(2, -2)*
B	*(3, -3)*	*(1, -1)*	*(1, -1)*
C	*(0, 0)*	*(1, -1)*	*(2, -2)*

A **saddle point** of a 2 player zero-sum game is defined to be an outcome where the payoff of Player 1 is maximized along its column but minimized along its row.

Problem 3.5. *Give an analogous definition of a saddle point in terms of Player 2's payoffs.*

Problem 3.6. *Find the saddle point of the previous matrix game. Verify that it is also a Nash equilibrium.*

Theorem 1. *The saddle points of a 2 player zero-sum game are exactly its pure Nash equilibria.*

Problem 3.7. *Prove Theorem 1.*

Problem 3.8. *Find the (pure) Nash equilibria of the following zero-sum game by finding its saddle points.*

Strategy	*D*	*E*	*F*	*G*
A	*(-4, 4)*	*(0, 0)*	*(3, -3)*	*(4, -4)*
B	*(-6, 6)*	*(1, -1)*	*(2, -2)*	*(3, -3)*
C	*(-3, 3)*	*(0, 0)*	*(-1, 1)*	*(-2, 2)*

4 Mixed Strategies

Action	L	R
T	(0, 3)	(3, 0)
B	(2, 1)	(1, 2)

Consider a game with many rounds, where each round has the above payoff matrix. Notice that neither player has a dominant strategy. How should Player 1 approach this game? Should they pick action T in every round and just hope Player 2 picks action R? Should they alternate actions every round, starting with B and then switching every time? What are the downsides of these approaches?

In the absence of a dominant strategy, the issue with Player 1 making a predictable choice in strategies is that Player 2 can learn and react to these choices. If Player 2 knows for certain that Player 1 will pick action T or B in any round, they will pick action L or R, respectively, which will always minimize Player 1's payoff (either 0 or 1). Therefore, as Player 1, we want to introduce some element of *chance* into our strategy.

For a player in a game with a choice of n actions, we define a **mixed strategy** as a probability distribution amongst the possible actions, $S = (p_1, p_2, \ldots, p_n)$, where each p_i represents the probability of choosing action i. For it to be a valid probability distribution, we require $0 \leq p_i \leq 1$ for all actions i, and $p_1 + p_2 + \cdots + p_n = 1$. If for any action i, $p_i = 1$, this is known as a **pure strategy**. Let's see why one might use a mixed strategy.

Problem 4.1. *Again, consider the game with many rounds, where each round has the below payoff matrix. Suppose that Player 1 plays the mixed strategy $S_1 = (0.1, 0.9)$ in every round, and Player 2 knows this. What will Player 2's optimal strategy be? What will Player 1's expected payoff per round be in this case?*

Action	*L*	*R*
T	*(0, 3)*	*(3, 0)*
B	*(2, 1)*	*(1, 2)*

Notice that, by using a mixed strategy, Player 1's expected payoff is more than if they picked either action predictably! In this vein, both players will want to employ mixed strategies to maximize their expected payoff.

Problem 4.2. *Again, consider the game with the above payoff matrix. Suppose Player 1 uses mixed strategy $S_1 = (p_1, p_2)$ and Player 2 uses mixed strategy $S_2 = (q_1, q_2)$. What is the expected payoff of each player? Your answer should be in terms of p_1, p_2, q_1, q_2.*

Now the question arises: what is the optimal mixed strategy for Player 1, given that Player 2 will know their strategy? The idea is to choose a mixed strategy that maximizes expected payoff no matter what Player 2's action is.

Problem 4.3. *Again, consider the game with the below payoff matrix. Suppose Player 1 uses mixed strategy* $S_1 = (p_1, p_2)$*. What is Player 1's payoff when Player 2 uses strategy* $S_2 = (1, 0)$*? What about when Player 2 uses strategy* $S_2' = (0, 1)$*? Find the values of* p_1 *and* p_2 *that maximize the minimum of these two payoffs. (Hint: think about when the payoffs are equal)*

Action	*L*	*R*
T	*(0, 3)*	*(3, 0)*
B	*(2, 1)*	*(1, 2)*

A **mixed Nash equilibrium** is a situation when no player can improve the expected payoff of their mixed strategy, given that the other players' mixed strategies are fixed.

Theorem 2 (Nash's Theorem)**.** *Every game with a finite number of players and a finite number of actions for each player has a mixed Nash equilibrium.*

The proof of this (perhaps surprising) theorem is beyond our scope.

Problem 4.4. *Find the mixed Nash equilibrium of the below payoff matrix. What is the expected payoff for each player? (Hint: you have already found Player 1's optimal mixed strategy)*

Action	*L*	*R*
T	*(0, 3)*	*(3, 0)*
B	*(2, 1)*	*(1, 2)*

Problem 4.5. *Find the mixed Nash equilibrium of rock, paper, scissors, given the following payoff matrix. What is the expected payoff for each player?*

Action	*Rock*	*Paper*	*Scissors*
Rock	*(0, 0)*	*(-1, 1)*	*(1, -1)*
Paper	*(1, -1)*	*(0, 0)*	*(-1, 1)*
Scissors	*(-1, 1)*	*(1, -1)*	*(0, 0)*

Problem 4.6. *Consider the game of chicken, where two players drive speedily toward each other, and each player either swerves or continues driving straight. If one player swerves and the other doesn't, the player who continued to drive straight wins while the other is labelled "chicken". If both players swerve, it is considered a tie. If neither player swerves, it is also considered a tie, except both players end up in a car crash and suffer from injuries (and sky high medical bills). Consider the following payoff matrix for chicken.*

Action	*Swerve*	*Drive Straight*
Swerve	*(0, 0)*	*(-1, 1)*
Drive Straight	*(1, -1)*	*(-4, -4)*

Find the Nash equilibria. (Hint: look for both pure and mixed) What are the players' expected payoffs in each case?

Problem 4.7. *Find all the Nash equilibria to the game with the following payoff matrix. (Hint: there are more than you can count)*

Action	*L*	*R*
T	*(3, -3)*	*(0, 0)*
M	*(0, 0)*	*(3, -3)*
B	*(2, -2)*	*(2, -2)*

Problem 4.8 (Challenge)**.** *Think back to Nash's Theorem. Why do we need the assumption that each player has a finite number of actions? Construct an infinite 2 player game without any Nash equilibria (pure or mixed).*