OLGA RADKO MATH CIRCLE, WINTER 2024: ADVANCED 3

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Worksheet :

Throughout this worksheet $\mathbb{F}$ is a field.

A vector space over a field $\mathbb{F}$, is given by a set V and two operations:

(1) vector addition $(+)$: an operation between two elements in V.
(2) Scalar multiplication $(\cdot)$: An operation between an element of $\mathbb{F}$ and an element of V.

Satisfying the following properties, for any u, v, w in V and α, β in $\mathbb{F}$:

(1) Associativity: $u + (v + w) = (u + v) + w$
(2) Commutativity: $u + v = v + u$
(3) Identity element: There exists an element 0_V in V, such that $u + 0_V = u$
(4) Inverse element: There exists an element $-u$ in V, such that $(-u) + u = 0_V$
(5) Compatibility of scalar multiplication and field multiplication: $(\alpha\beta)u = \alpha(\beta u)$
(6) Identity element of scalar multiplication: $1v = v$
(7) Distributivity of scalar multiplication with respect to vector addition: $\alpha(u + v) = \alpha u + \alpha v$
(8) Distributivity of scalar multiplication with respect to field addition: $(\alpha + \beta)v = \alpha v + \beta v$

Remember that $\mathbb{F}^n$ is also called the *affine space* $\mathbb{A}^n_{\mathbb{F}}$.

We will label the coordinates in affine space $\mathbb{A}^n_{\mathbb{F}}$:

$$(x_1, x_2, \ldots, x_n).$$

There are two operations that can be performed with points in affine space:

- Multiplication by scalars: If $c \in \mathbb{F}$ and $v = (x_1, x_2, \ldots, x_n) \in \mathbb{A}^n_{\mathbb{F}}$, the scalar multiplication is defined as
$$cv := (cx_1, cx_2, \ldots, cx_n).$$
- Addition of points: If $v = (x_1, x_2, \ldots, x_n) \in \mathbb{A}^n_{\mathbb{F}}$ and $w = (y_1, y_2, \ldots, y_n) \in \mathbb{A}^n_{\mathbb{F}}$, the addition is defined as
$$v + w := (x_1 + y_1, \ldots, x_n + y_n).$$

Problem 7.1:

Show that $\mathbb{A}^n_{\mathbb{F}}$ is a vector space over $\mathbb{F}$.

Solution 7.1:

Problem 7.2:

Show that the following are vector spaces:

(1) $\mathbb{C}$ over $\mathbb{R}$, where vector addition and scalar multiplication are the same as in $\mathbb{C}$.
(2) $\mathbb{R}$ over $\mathbb{Q}$, where vector addition and scalar multiplication are the same as in $\mathbb{R}$.
(3) $\mathbb{Q}[x]$ over $\mathbb{Q}$, where vector addition and scalar multiplication are the same as in $\mathbb{Q}[x]$.
(4) $\mathbb{F}_4$ over $\mathbb{F}_2$, where vector addition and scalar multiplication are the same as in $\mathbb{F}_4$.

Solution 7.2:

Let V be a vector space over $\mathbb{F}$. Let v be in V and α in $\mathbb{F}$.

Problem 7.3:

Show directly from the definitions that:

- $\alpha \cdot 0_V = 0_V$
- $0 \cdot v = 0_V$
- If $\alpha \neq 0$ and $v \neq 0_v$, then $\alpha \cdot v \neq 0_v$.

Solution 7.3:

Problem 7.4:

Show that any finite field of characteristic p is a vector space over $\mathbb{F}_p$.

Solution 7.4:

Let V be a vector space over $\mathbb{F}$.

A set $S \subseteq V$ is said to be a generating set if any element in V is a linear combination of elements in S. In other words, given any element v in V, there exist some elements $s_1, \ldots, s_r$ in S, and scalars $\alpha_1, \ldots, \alpha_r$ in $\mathbb{F}$, such that:

$$v = \alpha_1 s_1 + \ldots \alpha_r s_r.$$

Problem 7.5:

Find generating sets of the following vector spaces.

- $\mathbb{A}^2_{\mathbb{F}_3}$ as a vector space over $\mathbb{F}_3$.
- $\mathbb{A}^2_{\mathbb{F}_4}$ as a vector space over $\mathbb{F}_2$.
- $\mathbb{F}[x]$ as a vector space over $\mathbb{F}$.

Solution 7.5:

Let V be a vector space over $\mathbb{F}$.

A set $S \subseteq V$ is called a basis of V, if every non-zero element in V is a linear combination of elements in S in a unique way, up to zero scalars. In other words, for any non-zero element v in V, there there exist some elements $s_1, \ldots, s_r$ in S, and non-zero scalars $\alpha_1, \ldots, \alpha_r$ in $\mathbb{F}$, such that:

$$v = \alpha_1 s_1 + \ldots \alpha_r s_r.$$

And there are no other ways to choose s_i and α_i, except for permuting the elements.

A vector space is called finite dimensional if it has a basis with finitely many elements.

Problem 7.6:

Find a basis for:

- $\mathbb{A}^2_{\mathbb{F}_3}$ as a vector space over $\mathbb{F}_3$.
- $\mathbb{A}^2_{\mathbb{F}_4}$ as a vector space over $\mathbb{F}_2$.

Could you find a basis with fewer elements?

Solution 7.6:

Problem 7.7:

Find a basis for the following vector spaces:

(1) $\mathbb{A}^3_{\mathbb{F}_q}$ over $\mathbb{F}_q$.
(2) $\mathbb{F}_4$ over $\mathbb{F}_2$.
(3) $\mathbb{F}_8$ over $\mathbb{F}_2$.

Solution 7.7:

Problem 7.8:

Which of the following are generating sets?

(1) $\{(1,1),(2,2)\}$ in $\mathbb{A}^2_{\mathbb{F}_5}$ over $\mathbb{F}_q$.
(2) $\{(0,1),(1,0)\}$ in $\mathbb{A}^2_{\mathbb{F}_3}$ over $\mathbb{F}_3$.
(3) $\{(0,1),(1,0)\}$ $\mathbb{A}^2_{\mathbb{F}_4}$ over $\mathbb{F}_2$.
(4) $\{0,1\}$ $\mathbb{F}_2$ over $\mathbb{F}_2$.
(5) $\{1\}$ $\mathbb{F}_5$ over $\mathbb{F}_5$

Which of them are bases?

Solution 7.8:

For the following exercises you may use without giving a proof the following two facts:

- Any vector space has a basis
- If a vector space has a finite basis, then all of its basis have the same amount of elements.

The dimension of a finite dimensional vector space is defined as the number of elements in a basis of it.

Problem 7.9:

Compute the dimension of $\mathbb{A}^n_{\mathbb{F}_q}$ as a vector space over $\mathbb{F}_q$.

How many elements does a d-dimensional vector space over $\mathbb{F}_q$ have?

Solution 7.9:

If in the definition of a vector space we replace $\mathbb{F}$ with a ring R, we obtain what is called a module over R.

Problem 7.10:

Show that $\mathbb{Z}/m\mathbb{Z}$ is a module over $\mathbb{Z}$, where scalar product is given by the product in $\mathbb{Z}/m\mathbb{Z}$ after restricting $\mathbb{Z}$ modulo m.

Solution 7.10:

Problem 7.11:

Show that some modules do not have a basis.

Solution 7.11:

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