Permutations Part II - Partial Orderings

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Advanced 1 - Winter 2024

1 Definitions and Examples

Definition 1 *A* ***partially ordered set*** *(poset for short)* $(X, \preceq)$ *is a set* X *together with a binary relation* $\preceq$ *on the elements of* X*—that is, for each pair* $x, y \in X$*, either* $x \preceq y$ *or* $x \npreceq y$*—such that for any* $x, y, z \in X$*,*

- *(Reflexivity)* $x \preceq x$
- *(Symmetry) If* $x \preceq y$ *and* $y \preceq x$*, then* $x = y$*.*
- *(Transitivity) If* $x \preceq y$ *and* $y \preceq z$*, then* $x \preceq z$*.*

The relation $\preceq$ *is called the* ***partial ordering*** *on* X*. If* $x \preceq y$ *but* $x \neq y$*, we write* $x \prec y$*, and say that* x ***precedes*** y *in this case.*

Problem 1 *Let* X_n *be the set of integers* $\{1, 2, \ldots, n\}$ *for some* $n > 1$*. Decide whether or not the following relations are valid partial orderings for* X_n*. If not, which part of Definition 1 do they violate?*

- $x \preceq y$ *if* $x \leq y$ *(as integers).*

  Solution: Yes, partial order.
- $x \preceq y$ *if* $x < y$ *(as integers).*

  Solution: No, not reflexive.
- $x \preceq y$ *if* $x \geq y$ *(as integers).*

  Solution: Yes, partial order.
- $x \preceq y$ *if* $x = y$*.*

  Solution: Yes, partial order.
- $x \preceq y$ *if* $x \neq y$*.*

  Solution: No, fails all three.
- $x \preceq y$ *for any* x *and* y*.*

  Solution: No, not symmetric.
- $x \preceq y$ *if* y *is divisible by* x*.*

  Solution: Yes, partial order.

Problem 2 *The n-dimensional **boolean lattice** is the set B_n of all ordered n-tuples of 0's and 1's. Given two ordered n-tuples $x, y \in B_n$, we say $x \preceq y$ if every coordinate of x is less than or equal to the corresponding coordinate of y. For instance, $(0,0,1) \preceq (0,1,1)$ in B_3, but $(0,0,1) \not\preceq (0,1,0)$ in B_3. Show that $(B_n, \preceq)$ as defined is a poset.*

Solution: This follows from using the properties of $\leq$ coordinate-wise.

As with all kinds of sets that we've studied, we will call two posets *isomorphic* if they are the same, in the sense that the relations are the same. More formally,

Definition 2 *Two posets $(X, \preceq)$ and $(X', \preceq')$ are **isomorphic** if there is a relabeling of the elements of X as elements of X' such that the relation $\preceq$ is the same relation as $\preceq'$.*

An easy way to check whether two (finite) posets are isomorphic is to list out every pair of elements in each poset that are related to each other.

Problem 3 *Let $(B_2, \preceq)$ be defined as in Problem 2. Write down every pair of ordered pairs $x, y \in B_2$ where $x \preceq y$. Then do the same for $X_4 = \{1, 2, 3, 4\}$ using any of the valid partial orderings $\preceq$ from Problem 1. Are these isomorphic? Are any of those partial orderings from Problem 1 isomorphic to $(B_2, \preceq)$?*

Solution: See the comparability graphs on the next page.

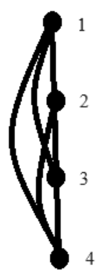
2 Graphing Posets

As you've seen from Problem 3, listing all the relations can take a lot of time (and those examples only had 4 elements each). To save the trouble of comparing two lists of relations, we introduce a way to visualize them. Recall the definitions and conventions of *directed graphs* from last week.

Definition 3 *Let $(X, \preceq)$ be a poset. Its **comparability graph** is the directed graph whose vertices are the elements of X, and for every $x \prec y$, there is a directed edge from x to y.*

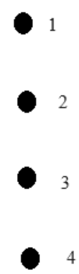
Problem 4 *For each example of a poset from Problems 1 and 2, draw their comparability graphs.*

Solution: Every graph in these solutions is directed top to bottom.

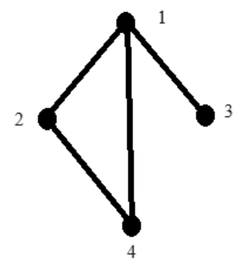


(X_4, <=)

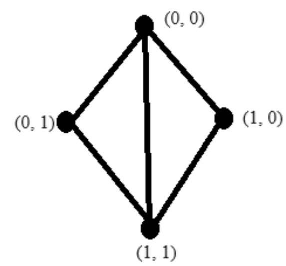
(X_4, >=)



(X_4, =)



(X_4, divisible by)



(B_2, above relation)

Problem 5 *Directed graphs are **isomorphic** if we can relabel the vertices of one graph with the vertices of the other so that the edges are the same (as directed edges). Show that two posets are isomorphic if and only if their comparability graphs are isomorphic.*

Solution: Follows from the definitions.

As the saying goes, a picture's worth a thousand words, and the comparability graph might certainly be easier to look at than a list. However, it didn't really solve our problem of having to write down every relation (they're just edges now). To make our lives easier from now on, let us clean up the comparability graph.

Definition 4 *Given a poset $(X, \preceq)$ and $x, y \in X$, we say that y* ***covers*** *x if $x \prec y$ and there is no $z \in X$ such that $x \prec z \prec y$. If y covers x, we call the relation $x \prec y$ a covering relation.*

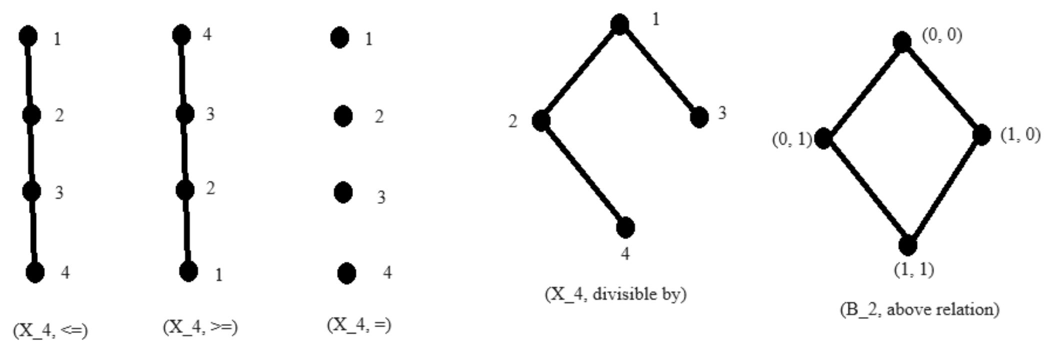
Problem 6 *Go through your lists of relations from Problem 3. Which ones are covering relations?*

Solution: The ones shown in the below Hasse diagrams.

Definition 5 *Let $(X, \preceq)$ be a poset. Its* ***Hasse diagram*** *is the directed graph whose vertices are the elements of X, and for every covering relation $x \prec y$, there is a directed edge from x to y.*

Problem 7 *Given a comparability graph for a poset, how do you find its Hasse diagram? And vice versa, given a Hasse diagram, how do you find the comparability graph? For examples, try this for the diagrams you found in Problem 4.*

Solution: Given a Hasse diagram, we add extra edges any time there are two or more arrows in a row. From a comparability graph, we just delete the extra edges. The following are the Hasse diagrams.



Problem 8 *Show that two posets are isomorphic if and only if their Hasse diagrams are isomorphic. Therefore it suffices to just draw the Hasse diagram.*

Solution: A sequence of covering relations is the same as a relation in a poset by transitivity. (Alternatively, this can be proven on the level of directed graphs.)

Problem 9 *Last week, we drew Cayley graphs of two different groups with four elements. Compare those Cayley graphs to the four-element Hasse diagrams that you've just drawn. What are the similarities? Make a guess as to what's going on here.*

3 The Weak Order of Permutations

We now return to last week's topic of permutations—for definitions and conventions (including bracket notation and cycle notation), see that handout.

We'll use the following terminology. Given a permutation $[a_1 \dots a_n]$, if two adjacent numbers are in the order $a_i > a_{i+1}$ and we make a new permutation switching just the two numbers a_i, a_{i+1}, we'll call that a *sorting move.* [1] If σ can be turned into τ by a sequence of sorting moves, we say that τ is *more sorted* than σ, and we write $\tau \preceq \sigma$. For example, the permutations [213] in S_3 is more sorted than [321], because we can switch the first two numbers, and then the last two, to get from [321] to [213].

Problem 10 *For the following σ and τ, determine whether $\sigma \preceq \tau$, $\tau \preceq \sigma$, or neither.*

- $\sigma = [123], \tau = [312]$

  Solution: $\sigma \preceq \tau$.

- $\sigma = [231], \tau = [132]$

  Solution: Neither.

- $\sigma = [4231], \tau = [1324]$

  Solution: $\tau \preceq \sigma$.

- $\sigma = [2134], \tau = [1243]$

  Solution: Neither.

- $\sigma = [54321], \tau = [51432]$

  Solution: $\tau \preceq \sigma$.

Problem 11 *Show that $\preceq$ is a partial order on S_n.*

Solution: $\sigma \preceq \sigma$ by an empty set of sorting moves. Since sorting moves don't go backwards, we also have symmetry. Finally, transitivity corresponds to concatenating two sequences of sorting moves.

[1] Sorting moves are the basis for a certain sorting algorithm, called bubblesort. In that context, they are more commonly called "inversions". We won't be handling lots of sorting today, but we may revisit this topic in future weeks.

Problem 12 *What are the covering relations for* $\preceq$*?*

Solution: Those related by a single sorting move.

Problem 13 *Draw the comparability graph and Hasse diagram for* $(S_3, \preceq)$*. How do these compare to the Cayley graphs we drew last week?*

Solution: Hexagon (see last week's worksheet).

Problem 14 *Draw the Hasse diagram for* $(S_4, \preceq)$*, and compare it to the Cayley graph.*

Solution: Permutohedron of order 4.

Problem 15 *Explain why the Hasse diagram for $(S_n, \preceq)$ gives the same permutohedron shape that the Cayley graph does.*

Solution: Many different arguments. Most will rely on the fact that sorting moves are analogous to adjacent transpositions in the correct way.

All the facts we showed last week about the permutohedron hold for these Hasse diagrams. But Hasse diagrams have a natural direction that we didn't have before, and we can describe some permutations by "how sorted" they are. We may revisit this too in a later week, but for now let's show the following two properties.

Problem 16 *Does S_n have an element that is the most sorted? That is, is there $\sigma \in S_n$ such that $\sigma \preceq \tau$ for all $\tau \in S_n$? If so, find σ. If not, explain why.*

Solution: The identity element $[12\ldots n]$ is minimal.

Problem 17 *Does S_n have an element that is the least sorted? That is, is there $\sigma \in S_n$ such that $\tau \preceq \sigma$ for all $\tau \in S_n$? If so, find σ. If not, explain why.*

Solution: The reflection of the identity $[n\ldots 21]$ is maximal.

Problem 18 *(Bonus) So far, we've been studying the weak order of permutations, which begs the question of what the strong order of permutations is. For this, we have to fix a set of generators for S_n (as a group), and for convenience, we usually use the set of adjacent transpositions. We define the **strong Bruhat order** on S_n as follows: $\sigma \preceq \tau$ if, when both σ and τ are written as a product of adjacent transpositions, the expression for σ is a substring of the expression for τ. Show this is a partial ordering, and draw the Hasse diagrams of some small S_n (say, S_3 and S_4). How do these compare to the permutohedra?*