Logic Problems and Essential Formulas

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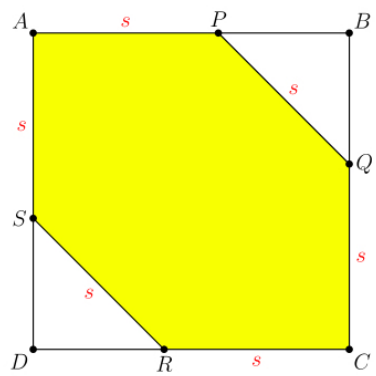
1 Warm Up

Square $ABCD$ has side length 1. Points P, Q, R, and S each lie on a side of $ABCD$ such that $APQCRS$ is an equilateral convex hexagon with side length s. What is s?



A) $\frac{\sqrt{2}}{3}$ B) $\frac{1}{2}$ C) $2-\sqrt{2}$ D) $1-\frac{\sqrt{2}}{4}$ E) $\frac{2}{3}$

There are two values of a for which the equation $4x^2+ax+8x+9=0$ has only one solution for x. What is the sum of those values of a?

(A) -16 (B) -8 (C) 0 (D) 8 (E) 20

2 Logic Problems

Logic problems test your ability to think critically and quickly. Generally, logic problems are the ones that can give you an “Aha!” moment when you find out the trick. To do this, you just have to be organized.

2.1 There are quick solutions!

Some logic problems will present you with a straightforward, but long and tiring solution. But the fastest way to solve them is to find the trick behind the problem. So, let’s start off with an example of a long and tiring, but unnecessary solution.

41 players sign up for a 32 person Best-of-1 ping pong competition, where if you lose a game, you’re out. To decrease the number of players, 18 of them are randomly chosen to play against each other in pairs, with the 9 winners moving on. Then, the remaining 32 players face off in an elimination bracket. In each round, players go against a random opponent, with the winner progressing and the loser out. This continues until a single winner is left. In total, how many games are played?

2.2 Repetition

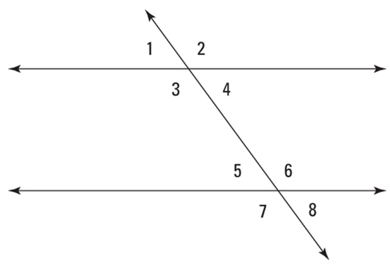
Another type of logic problem presents you with repetition. Sometimes, the problem will ask you to do the same things tens of times, or at least more than you should, and condensing what you’re repeating makes it much faster.

Jenny has 3 different books, a dictionary, thesaurus, and English textbook. She puts them on a shelf in order, from left to right: dictionary, thesaurus, and textbook. For 24 days, at 1 pm and 3 pm each day, she switches the places of the 2 leftmost books. At 2 pm each day, she switches the place of the 2 rightmost books. At 1:30 pm on the thirteenth day, what book is in the leftmost position?

3 2D Geometry Formulas

3.1 Angles

Using parallel lines, we can examine pairs of angles. When a line passes through a pair of parallel lines, it is called a transversal. A transversal creates eight angles, as shown below.



From the above diagram, we can find special pairs of angles:
- Corresponding angles, like$\angle 1$ and $\angle 5$, are **equal**.
- Alternate Interior angles, like $\angle 3$ and $\angle 6$, are **equal**
- Alternate Exterior angles, like $\angle 1$ and $\angle 8$, are **equal**
- Consecutive Interior angles, like $\angle 3$ and $\angle 5$, are **supplementary**, meaning they add up to 180°

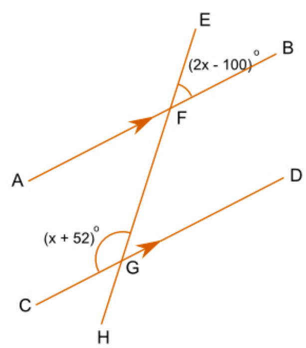
And, as always, a straight line is just a 180°. This means that if a straight line is divided into various unknown angles, you can add them all up and set them equal to 180 to solve for the angle measures.

For the sum of interior angles for any polygon, you can use the formula $180(n-2)$, where n is the number of sides on the polygon. The sum of **exterior** angles of a polygon is just $\frac{180(n-2)}{n}$.

3.1.1 Angles Practice

Find the sum of interior angles of a hexagon. Then find the sum of exterior angles.

AB and CD are parallel lines and EH is a transversal. The size of angle EFB is $(2x - 100)$ and the size of angle CGF is $(x + 52)$. What is the actual size of the $\angle EFB$?



3.2 Triangles

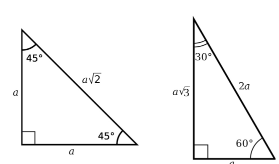
The basic area for a triangle is $\frac{b \cdot h}{2}$, where b is the base and h is the height. We have already looked at Heron's Area Formula. To use the formula, we need values where s is the semi-perimeter (half of the perimeter), and a, b, and c are side lengths of the triangle.

In Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$

We have also looked at Pythagorean Theorem for right triangles: $a^2 + b^2 = c^2$. Remember this only works for **right triangles**, and c is always the hypotenuse while a and b are the legs.

It can help to know some Pythagorean triples. Can you name some of them?

We can also remember our special right triangles, the $30 - 60 - 90$ triangle and the $45 - 45 - 90$, or right isosceles, triangle. These values refer to the angle measures of each triangle. If we see all of these angles, we know that the side lengths follow the below ratios:

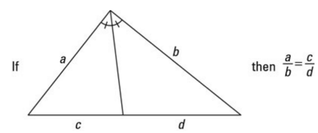


For triangles in a graph with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3),

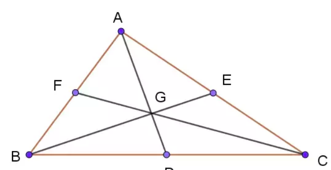
$$\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

This is helpful when it's hard to find the length of the sides of the triangle.

Angle Bisector Theorem is all about the angle bisector, which splits an angle in half. The bisector can split a triangle into very easy ratios, as shown below:



You may also find it helpful to know about medians of triangles. A median simply splits the opposite side in half. A triangle has three medians that intersect at a centroid, as shown below.



Using the medians, we can form a very useful ratio. The medians are split in $2 : 1$ ratios. This just means that CG=$2GF$, BG=$2GE$, and AG=$2GD$. Each of the six triangles formed by the intersecting medians also all have the same areas!

3.2.1 Triangles Practice

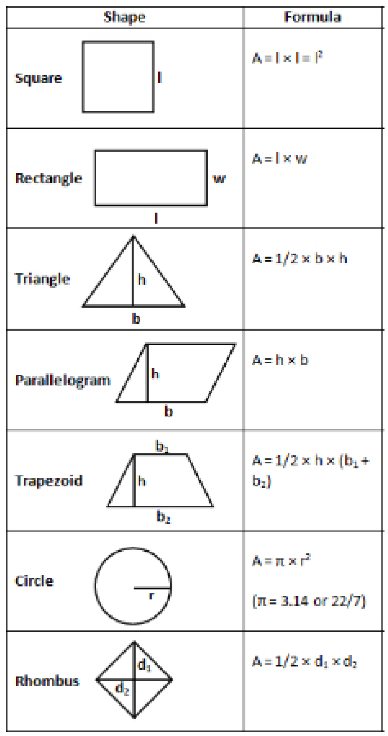
Evaluate the area of the triangle with points $(2, 5)$, $(7, 3)$, and $(3, 0)$.

Find the leg lengths of a $30 - 60 - 90$ triangle with hypotenuse of 6. Then find the area of this triangle.

Find the area of a triangle with side lengths 13, 14, and 10.

3.3 Other 2D Areas

Geometry is very very important on the AMC! It is critical to know the below area formulas for basic shapes. These area formulas can also be utilized later to find volume of 3D shapes, as we will be doing next week!



Shape	Formula
Square	$A = l \times l = l^2$
Rectangle	$A = l \times w$
Triangle	$A = 1/2 \times b \times h$
Parallelogram	$A = h \times b$
Trapezoid	$A = 1/2 \times h \times (b_1 + b_2)$
Circle	$A = \pi \times r^2$ (π = 3.14 or 22/7)
Rhombus	$A = 1/2 \times d_1 \times d_2$

A couple of quick things to remember are that:

1. A rhombus has four equal sides, two equal obtuse angles, and two equal acute angles.

2. A square is a rectangle with all sides of equal length. A square is also a rhombus of all angles equal length. This means that a set of squares is the intersection of the sets of rectangles and rhombi.

3. For all the shapes except the circle, the perimeter is just the sum of the side lengths. For a circle, the perimeter is $2\pi r$.

4. There are many other questions that can be asked about circles (including in our practice problems today)! We went over these formulas only two weeks ago, so you should remember them. Try to see how you do on the practice problems!

4 Practice

The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

(A) 7 **(B)** 8 **(C)** 16 **(D)** 17 **(E)** 18

Mr. Earl E. Bird gets up every day at 8 : 00AM to go to work. If he drives at an average speed of 40 miles per hour, he will be late by 3 minutes. If he drives at an average speed of 60 miles per hour, he will be early by 3 minutes. How many miles per hour does Mr. Bird need to drive to get to work exactly on time?

(A) 45 **(B)** 48 **(C)** 50 **(D)** 55 **(E)** 58

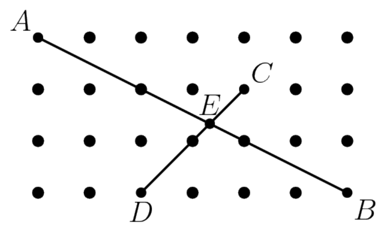
Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least 6 miles away," Bob replied, "We are at most 5 miles away." Charlie then remarked, "Actually the nearest town is at most 4 miles away." It turned out that none of the three statements were true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d?

(A) $(0, 4)$ **(B)** $(4, 5)$ **(C)** $(4, 6)$ **(D)** $(5, 6)$ **(E)** $(5, \infty)$

A triangle has side lengths 10, 10, and 12. A rectangle has width 4 and area equal to the area of the triangle. What is the perimeter of this rectangle?

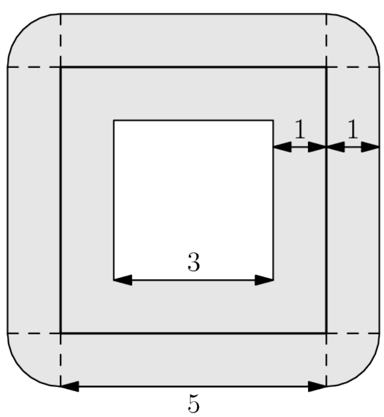
(A) 16 **(B)** 24 **(C)** 28 **(D)** 32 **(E)** 36

The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE.



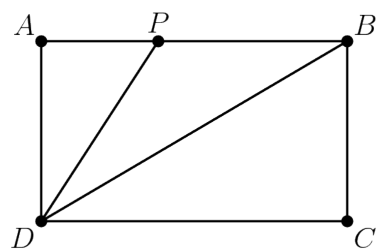
(A) $\frac{4\sqrt{5}}{3}$ **(B)** $\frac{5\sqrt{5}}{3}$ **(C)** $\frac{12\sqrt{5}}{7}$ **(D)** $2\sqrt{5}$ **(E)** $\frac{5\sqrt{65}}{9}$

Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk, expressed in square kilometers and rounded to the nearest whole number?



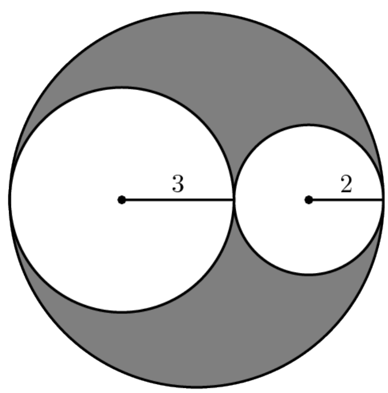
(A)24 **(B)** 27 **(C)** 39 **(D)** 40 **(E)** 42

In rectangle $ABCD$, $AD = 1$, P is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. What is the perimeter of $\triangle BDP$?



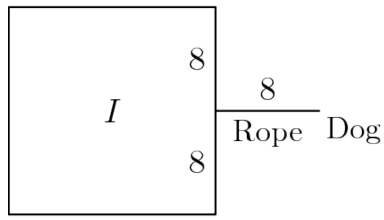
(A) $3 + \frac{\sqrt{3}}{3}$ **(B)** $2 + \frac{4\sqrt{3}}{3}$ **(C)** $2 + 2\sqrt{2}$ **(D)** $\frac{3+3\sqrt{5}}{2}$ **(E)** $2 + \frac{5\sqrt{3}}{3}$

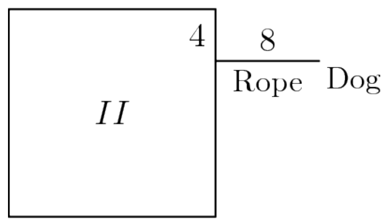
Circles of radius 2 and 3 are externally tangent and are circumscribed by a third circle, as shown in the figure. Find the area of the shaded region.



(A) 3π (B) 4π (C) 6π (D) 9π (E) 12π

Rolly wishes to secure his dog with an 8-foot rope to a square shed that is 16 feet on each side. His preliminary drawings are shown.





Which of these arrangements give the dog the greater area to roam, and by how many square feet?

(A) I, by 8π **(B)** I, by 6π **(C)** II, by 4π **(D)** II, by 8π **(E)** II, by 10π