Permutations Part I - A Graphical Viewpoint

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Advanced 1 - Winter 2023

1 Reminders about Permutations

Before we start drawing, it will help to establish notation and properties of permutations. Recall that earlier this quarter, we showed that the set of permutations forms a group with composition, called the *symmetric group*. We recap these definitions below.

Definition 1 *A group is a set G together with a multiplication operation $\cdot$ such that*

- *(Associativity) For all $x, y, z \in G$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.*
- *(Identity) There is an element $e \in G$ such that for any $x \in G$, $e \cdot x = x \cdot e = x$. e is called the* ***identity element*** *of G.*
- *(Inverses) For all $x \in G$, there exists a $y \in G$ such that $x \cdot y = y \cdot x = e$.*

(Recall that we often write xy as a shorthand for $x \cdot y$.)

Definition 2 *The symmetric group S_n is the set of permutations σ of a set of n different symbols (which are commonly represented by $1, \dots, n$). The multiplication operation is as follows: given two permutations σ, τ of $1, \dots, n$, the permutation $\sigma\tau$ is the one that applies τ first, then σ, to the set of n symbols.*

Problem 1 *Let $\sigma, \tau \in S_4$ be the following permutations of four elements: σ sends 1 to 1, 2 to 3, 3 to 4, and 4 to 2, and τ sends 1 to 4, 2 to 3, 3 to 1, and 4 to 2. Find the permutation $\sigma\tau$.*

Solution: $\sigma\tau$ is the permutation 1 goes to 2, 2 goes to 4, 3 goes to 1, 4 goes to 3. (Caution: The permutation 1 goes to 4, 2 goes to 1, 3 goes to 2, 4 goes to 3 is $\tau\sigma$, not $\sigma\tau$.)

While we could continue describing permutations elementwise like we did in Problem 1, it will be helpful to introduce some notation.

Definition 3 *$[a_1 \dots a_n]$ denotes the permutation $\sigma \in S_n$ where σ sends i to a_i. For example, $[51432]$ is the permutation in S_5 that takes 1 to 5, 2 to 1, 3 to 4, 4 to 3, and 5 to 2.*

Problem 2 *Using our new notation, rewrite all three permutations $\sigma, \tau, \sigma\tau$ from Problem 1.*

Solution: $\sigma = [1342], \tau = [4312], \sigma\tau = [2413]$.

Problem 3 *Compute the following products:*

- [132][213]

  Solution: [312]

- [1234][1432]

  Solution: [1432]

- [51432][24315]

  Solution: [13452]

Problem 4 *In brackets notation (our current notation), what is the identity element of S_n for any given n?*

Solution: The identity permutation is the one that does nothing - in other words, $[123\dots n]$.

Problem 5 *Find the inverse of* [51432] *in* S_5. *In general, how would you find the inverse of a given permutation in brackets notation?*

Solution: $[51432]^{-1} = [25341]$. For each symbol, we put the position it occurs to make the inverse (for instance, 1 occurs in the second position, so the first symbol of the inverse is 2).

2 Cycles and Generators

While our previous notation is helpful for describing permutations element-wise, we have seen that group operations like multiplications and taking inverses can be quite clunky. We now introduce a new notation to denote permutations that will help us investigate the group structure.

Definition 4 *For $k \geq 2$, a k-cycle, denoted $(a_1 \ldots a_k)$ is a permutation that sends a_1 to a_2, a_2 to a_3, and so on, and a_k back to a_1, while keeping all other symbols that aren't $a_1 \ldots a_k$ the same. For example, the 3-cycle (234) sends 2 to 3, 3 to 4, 4 to 2, and 1 (and any other symbols present) to itself. 2-cycles in particular are also known as **transpositions**.*

Problem 6 *Find the products of the following cycles:*

- (12)(23)

  Solution: (123)

- (123)(234)

  Solution: (12)(34)

- (1234)(15)

  Solution: (15234)

- (123)(456)(132)(465)

  Solution: This is the identity.

- (56)(1789423)(56)

  Solution: (1789423)

- $(1789423)^{-1}(56)(1789423)(56)(1789423)(56)(1789423)^{-1}(56)$

  Solution: This is the identity.

In the last problem, we multiplied some examples of *disjoint* cycles (that is, cycles that don't have any numbers in common). Although permutations in general don't commute with each other, our answers suggest that disjoint cycles do commute.

Problem 7 *Evaluate the last three examples of Problem* 6 *by commuting disjoint cycles, if you didn't already. Can you explain why this works?*

Solution: It works because given two disjoint cycles, a symbol affected by one is not affected by the other, so overall the order doesn't matter.

Problem 8 *Explain how to write any permutation $\sigma \in S_n$ as a product of disjoint cycles. Such a product expression is called* ***cycle notation*** *for σ. (Hint: The identity element e can be written as an empty product. For any other permutation, start at an element that doesn't go to itself.)*

Solution: For any non-identity permutation, find the smallest number that doesn't go to itself. Whereever that number goes, find where the next number goes, and so on. This must repeat eventually since there's a finite number of symbols, so whenever it does that makes a cycle. Then we find the next smallest number that wasn't in that cycle, and repeat.

Problem 9 *How would you find the inverse of a given permutation in cycle notation? As an example, find the inverse of* $(1257)(348)$ *in* S_8. *Is this more or less convenient than with our previous bracket notation?*

Solution: $((1257)(348))^{-1} = (1752)(384)$. To invert each cycle, we keep the first number the same and flip all the other numbers.

Earlier this quarter, we described multiplication in certain groups by writing down their multiplication tables. But now that we have the groups S_n with $n!$ elements each, we would rapidly run out of paper trying to write down all these tables. Instead, we can describe multiplication in a group by describing multiplication of a few of its elements.

Definition 5 *Let S be a subset of a group G. S is said to* ***generate*** *G if any element of G can be given by multiplying together elements of S. (As before, the identity element e will be thought of as the empty product, which slightly justifies using that letter for the identity element.)*

Problem 10 *Earlier this quarter, we studied the following two examples of groups: the integers mod* 4 *with the operation + (top table below) and the rectangle symmetry group with the operation composition (bottom table below). For each of these groups, find subsets that generate them.*

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\circ$	e	r	f_1	f_2
e	e	r	f_1	f_2
r	r	e	f_2	f_1
f_1	f_1	f_2	e	r
f_2	f_2	f_1	e	r

Solution: $\{1\}$ and $\{3\}$ by themselves generate the integers mod 4, while $\{r, f_1\}$ and $\{r, f_2\}$ are generating sets for the rectangle symmetry group.

Problem 11 *Show that the set of cycles generates the symmetric group S_n.*

Solution: Any permutation can be written in cycle notation, so any permutation is a product of cycles.

Problem 12 *How many cycles, in total, does S_n have? Do you think that the set of cycles is a useful generating set?*

Solution: We count the k-cycles for each k and add them up. To form a k-cycle, we first choose n numbers. Since it is equivalent to start the cycle at any number, without loss of generality we start writing the cycle with its lowest number. There are $(k-1)!$ ways to arrange the other numbers, so there are $(k-1)!\binom{n}{k}$ many k-cycles. Therefore the total number of cycles is

$$\sum_{k=2}^{n}(k-1)!\binom{n}{k}$$

$(n-1)!$ by itself is very large, so this is not a useful generating set.

Problem 13 *Show that any k-cycle in S_n can be written as a product of $k-1$ transpositions. (Hint: Problem* 6 *has some examples of (small) cycles. Be inspired by them.)*

Solution: Let $(a_1 \dots a_k)$ be a k-cycle. Similarly to Problem 6, we have

$$(a_1 \dots a_k) = (a_1a_2)(a_2a_3)\dots(a_{k-1}a_k)$$

Problem 14 *Using Problems* 11 *and* 13*, show that the set of transpositions in S_n generate it. How many transpositions are there? Is this a more useful set?*

Solution: Since the order of the elements in a transposition doesn't matter, there are just $\binom{n}{2}$ transpositions in S_n. To write any permutation in terms of them, write it in cycle notation first, then break each cycle into permutations by Problem 13.

Problem 15 *Compute the following products of transpositions:*

- $(12)(23)(12)$

  Solution: (13)

- $(12)(23)(34)(23)(12)$

  Solution: (14)

- $(12)(23)(34)(45)(34)(23)(12)$

  Solution: (15)

Problem 16 *We define an* ***adjacent transposition*** *to be a transposition switching two adjacent numbers i and $i+1$. Show that the set of adjacent transpositions generate S_n. (Hint: By Problem 14 we can write any permutation as a product of transpositions. Use the previous problem to inspire you to write any transposition as the product of adjacent transpositions.) How many adjacent transpositions does S_n have?*

Solution: In general, the transposition (ij) can be written $(i, i+1)(i+1, i+2) \ldots (j-2, j-1)(j-1, j)(j-2, j-1) \ldots (i+1, i+2)(i, i+1)$. So we write any permutation as a product of transpositions by Problem 14, then each transposition as the product of adjacent transpositions in this way. There are only $n-1$ adjacent transpositions in S_n, so this is a more workable generating set.

Problem 17 *(Bonus) Find* ***two*** *elements that generate S_n for any n. (We won't be using these generators later, so do this problem whenever you have the time!)*

Solution: (12) and $(123 \ldots n)$ generate S_n. The proof uses tricks like the one shown in Problem 15.

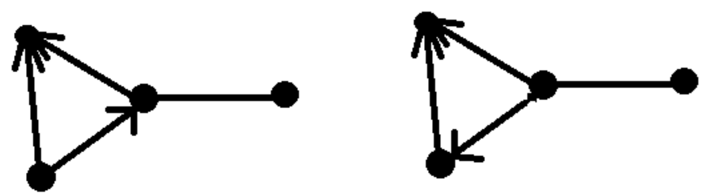
3 Visualizing Multiplication—Cayley Graphs

Given a group G with a generating set S, we can think about multiplying a lot of elements of S together. Each successive term $s_1, s_1 s_2, s_1 s_2 s_3, \ldots$ gives us an element of G, and together they give a path through G, of sorts. In order to draw this path, we'll represent G as a graph. Recall that a graph consists of a set of vertices V and a set of edges E between pairs of vertices.

Definition 6 *A* ***directed graph*** *(or digraph for short) is a set of vertices V along with a set of* ***directed edges*** *E—that is, edges $e \in E$ go from some vertex v to vertex w. (The opposite direction is considered a different directed edge.)*

Directed edges are typically drawn as arrows. If v and w in a directed graph have one edge going from v to w and one edge going from w to v, we often abbreviate it by drawing an undirected edge between them. For instance, these two are the same graph,

but these two are not—even though the shapes look the same, the right graph has a vertex with three edges coming *out* of it and the left graph does not.

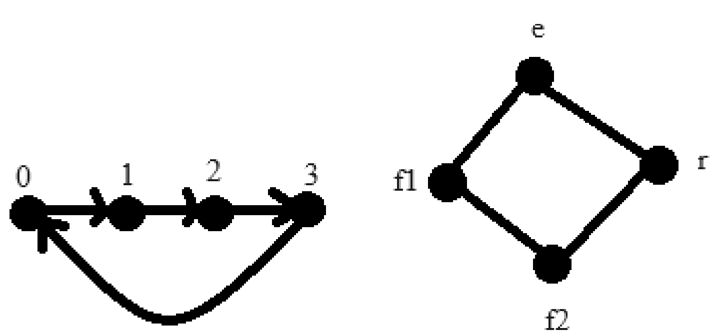


Definition 7 *Given a group G and a generating set S in G, its* ***Cayley graph*** *is the directed graph Γ given as follows:*

- *The vertices of Γ are the elements of G.*
- *For every $x \in G$ and every $s \in S$, Γ has a directed edge from x to xs.*

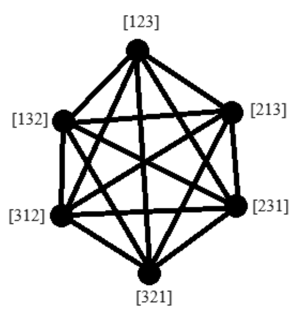
Problem 18 *For the groups and sets of generators you found in Problem* 10*, draw their Cayley graphs.*

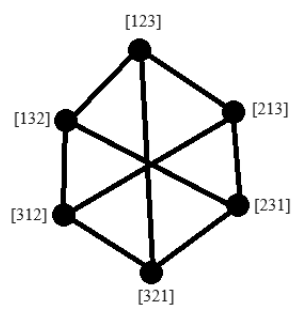
Solution:

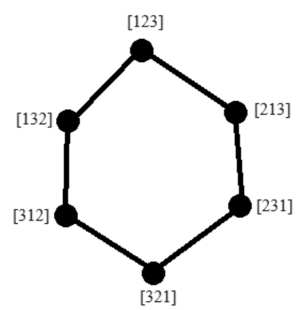


Problem 19 *Compare the three sets of generators for S_n that we found in the previous section: the cycles (Problem 11), the transpositions (Problem 14) and the adjacent transpositions (Problem 16). For S_2, the group of permutations of two symbols, is there a difference? Draw the Cayley graph for S_2.*

Problem 20 *Compare the three sets of generators for S_n that we found in the previous section: the cycles (Problem 11), the transpositions (Problem 14) and the adjacent transpositions (Problem 16). For S_3, the group of permutations of three symbols, is there a difference? Draw the Cayley graph given by each set of generators. Which one looks the cleanest?*







Problem 21 *Using the generating set of adjacent transpositions, draw the Cayley graph for S_4.*

Solution: See Wikipedia for a diagram. To draw it in a planar way, try your best.

These Cayley graphs for S_n (using the generating set of adjacent transpositions) are also called *permutohedra* [1] of order n, denoted P_n. Clearly, P_n will get a lot tougher to draw for higher n, so we restrict ourselves to describing their features. Specifically, we'll count and describe the *vertices, edges,* and *faces* of P_n. (Since P_n isn't planar, the notion of a face is a little more finicky. We'll consider any cycle that doesn't break into smaller cycles a face.)

Problem 22 *How many vertices does P_n have?*

Solution: Each vertex is a permutation, so there are $n!$ of them.

Problem 23 *How many edges does P_n have? (Hint: Think of what an edge corresponds to in terms of permutations.)*

Solution: There are $n-1$ adjacent transpositions in S_n, so each vertex of P_n has $n-1$ edges coming out of it. Each edge is between two vertices, so there are $(n-1)n!/2$ edges.

Problem 24 *Show that all faces of P_n are either quadrilaterals or hexagons. (Bonus) How many of each kind of face does P_n have?*

Solution: Each face consists of compositions of two transpositions. If the two transpositions overlap, they make a 3-cycle and that face is a copy of P_3 (a hexagon). If the two transpositions are disjoint, they commute and form the quadrilateral from Problem 18.

[1] This is one of several equivalent definitions for the permutohedron. We will encounter another one in the next section of this worksheet, and yet another one next week, so stay tuned.

4 Permutohedra in Space

Problem 25 *What dimensional shapes do P_1, P_2, and P_3 remind you of?*

Solution: A point (0D), a line segment (1D), and a hexagon (2D).

Unless you drew Problem 21 exceptionally well, realizing P_4 as a polyhedron might be a little tough. Instead, we'll show it's a *planar graph*. Recall that a planar graph is a graph that can be drawn in the plane without crossing edges, and that such a graph can be folded into a polyhedron in several ways, one of which is by adding a point at infinity.

Problem 26 *Show that P_4 is a planar graph, by drawing it below without crossing edges. Use this to sketch P_4 as a polyhedron in 3 dimensional space.*

By now, it should seem like P_n is an $(n-1)$-dimensional object. In order to prove this, we want a more systematic way of putting it into space.

Problem 27 *Consider the permutations of the coordinates of the point* $(1, 2, \ldots, n)$ *in* n*-dimensional space. Show that the polytope (general term for polyhedron in higher dimensions) with these vertices forms* P_n *with its edges.*

Problem 28 *Show that all the permutations of* $(1, 2, \ldots, n)$ *lie on the same hyperplane in* n*-dimensional space (and therefore form a* $(n-1)$*-dimensional object).*

Finally, P_3 (the hexagon) famously tessellates the plane (2-dimensional space). Also, P_2 (the line segment) trivially tessellates the line (1-dimensional space). In fact, this is true for all P_n, but is much harder to show for higher n. In the case of P_4, we can at least draw the picture:

Problem 29 *(Challenge) Show that* P_4 *tessellates* 3*-dimensional space.*