ORMC AMC 10/12 Group
Week 8: More Combinatorics

February 25, 2023

1 Warm-up Exercises

1. **(2008 AMC 12B #6)** Postman Pete has a pedometer to count his steps. The pedometer records up to 99999 steps, then flips over to 00000 on the next step. Pete plans to determine his mileage for a year. On January 1 Pete sets the pedometer to 00000. During the year, the pedometer flips from 99999 to 00000 forty-four times. On December 31 the pedometer reads 50000. Pete takes 1800 steps per mile. If Pete walked x miles during the year, find $\lfloor x \rfloor$.

2. **(2010 AMC 12A #19)** Each of 2010 boxes in a line contains a single red marble, and for $1 \leq k \leq 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

3. **(2012 AIME #1)** Find the number of positive integers with three not necessarily distinct digits, abc, with $a \neq 0$ and $c \neq 0$ such that both abc and cba are multiples of 4.

4. **(2006 AMC 10A #21)** How many four-digit positive integers have at least one digit that is a 2 or a 3?

5. **(Alcumus)** Henry's little brother has 8 identical stickers and 4 sheets of paper, each a different color. He puts all the stickers on the pieces of paper. How many ways are there for him to do this, if only the number of stickers on each sheet of paper matters?

2 Theorems/Techniques

There are a few basic counting techniques that you all are certainly familiar with, which are used frequently in the AMC:

- **Casework:** Breaking down a counting problem into several disjoint cases, and adding up the number of ways each case could occur
- **Complementary Counting:** When asked for the number of ways that a desired event could occur, instead count how many ways there are for it not to occur, and subtract this number from the total.
- **Overcounting:** Generally, this means we count some events more than once, either due to overlapping cases, or due to symmetries. In order to reduce back to the actual desired count, we can subtract away this over-counting (i.e. Principle of Inclusion-Exclusion), or divide it out (i.e. Burnside's Lemma)

Some well-known examples of overcounting are the Principle of Inclusion-Exclusion (PIE for short) and Burnside's Lemma, both of which have been discussed on previous worksheets. As a reminder, here are their statements:

2.1 PIE

When we have n cases $A_1, A_2, \ldots, A_n$ that possibly overlap, the number of ways that at least one of the n cases could occur is:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

2.2 Burnside's Lemma

Suppose we have a set S of configurations where symmetries (i.e. reflections, rotations) are counted as distinct, and a group G of symmetries. Then, the number of configurations where symmetries are considered indistinguishable (often denoted S/G), is:

$$|S/G| = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)| = \frac{1}{|G|} \sum_{c \in S} |\text{stab}(c)| = \frac{1}{|G|} |\{(g, c) \mid g(c) = c\}|$$

Where $\text{fix}(g)$ is the set of configurations which are fixed (unchanged) by a symmetry g, and $\text{stab}(c)$ is the set of symmetries which stabilize (do not change) a configuration c.

2.3 Distinguishability

Another important concept is **distinguishability**. This applies to the idea of placing items into categories. In general, these techniques apply to a problem if there is some way to view it as placing n balls into k boxes. There are 4 cases:

1. **The balls are distinguishable, and the boxes are distinguishable:** In this case, the total number of configurations is just $\boxed{k^n}$, since we have k choices for each of the n items.
2. **The balls are identical, but the boxes are distinguishable:** This is the **stars-and-bars** case. With n items and k categories, we have n stars and $k-1$ bars, giving $\boxed{\binom{n+k-1}{n}}$ configurations.
3. **The balls are distinguishable, but the boxes are identical:** This can be computed by using case 1, and applying Burnside's Lemma to the boxes, using all possible permutations of $\{1, 2, \ldots, k\}$ as the set of symmetries. The number of configurations is usually denoted $\left\{ {n \atop k} \right\}$ and is a Stirling number of the second kind.
4. **The balls are identical, and the boxes are identical:** Similar to 3, this can be computed using case 2 and Burnside's Lemma. These configurations are called partitions.

3 Exercises

1. **(2017 AMC 10B #13)** There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

2. **(Alcumus)** Bob and Meena play a two-person game which is won by the first person to accumulate at least 10 points. On each turn, there is a $\frac{2}{5}$ probability that Bob will get two points and Meena will lose one point. If that doesn't happen, then Meena gets two points and Bob loses a point. Meena is now ahead 9 to 6. What is the probability that Meena will win? Express your answer as a common fraction.

3. **(2008 AMC 12A #21)** A permutation $(a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$ is heavy-tailed if $a_1 + a_2 < a_4 + a_5$. What is the number of heavy-tailed permutations?

4. **(2011 AMC 12A #16)** Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

5. **(2008 AMC 12B #22)** A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

6. **(2005 AMC 12A #18)** Call a number prime-looking if it is composite but not divisible by $2, 3$, or 5. The three smallest prime-looking numbers are $49, 77$, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

7. **(2012 AMC 12B #16)** Amy, Beth, and Jo listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the girls, there is at least one song liked by those two girls but disliked by the third. In how many different ways is this possible?

8. **(2010 AMC 12A #16)** Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

9. **(Alcumus)** A machine randomly generates one of the five numbers 1, 2, …, 5 with equal likelihood. What is the probability that when Tsuni uses this machine to generate four numbers their product is divisible by 6 ? Express your answer as a common fraction.

10. **(Alcumus)** Two integers are relatively prime if they have no common factors other than 1 or -1. What is the probability that a positive integer less than or equal to 30 is relatively prime to 30? Express your answer as a common fraction.

11. Consider the case from warm-up #5, but suppose Henry's little brother refuses to place more than 3 stickers on the same sheet of paper. How many ways can he do this?

12. **(2009 AMC 10A #22)** Two cubical dice each have removable numbers 1 through 6. The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is 7?

13. **(Alcumus)** Rachel has two identical basil plants and an aloe plant. She also has two identical white lamps and two identical red lamps she can put each plant under (she can put more than one plant under a lamp, but each plant is under exactly one lamp). How many ways are there for Rachel to put her plants under her lamps?

14. **(2009 AMC 10A #24)** Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube?

15. **(2009 AMC 12B #21)** Ten women sit in 10 seats in a line. All of the 10 get up and then reseat

themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

16. **(Alcumus)** How many ways are there to put 4 balls in 3 boxes if the balls are distinguishable but the boxes are not? What if there are 5 balls?

17. **(2021 AMC 10A #20)** In how many ways can the sequence $1, 2, 3, 4, 5$ be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

18. **(Alcumus)** Matt's four cousins are coming to visit. There are four identical rooms that they can stay in. If any number of the cousins can stay in one room, how many different ways are there to put the cousins in the rooms?

19. **(2010 AMC 12A #15)** A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

20. **(2010 AMC 12B #17)** The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

21. **(2012 AMC 12B #18)** Let $(a_1, a_2, \ldots, a_{10})$ be a list of the first 10 positive integers such that for each $2 \le i \le 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such lists are there?