3D Geometry

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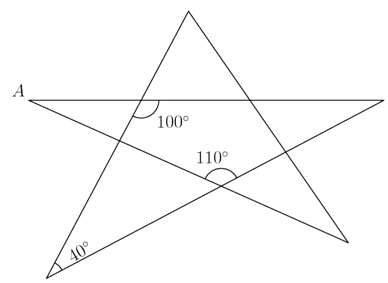
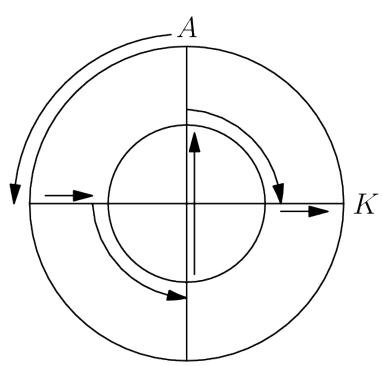
1 Warm Up

The degree measure of angle A is



(A) 20 (B) 30 (C) 35 (D) 40 (E) 45

Two circles that share the same center have radii 10 meters and 20 meters. An aardvark runs along the path shown, starting at A and ending at K. How many meters does the aardvark run?



(A) $10\pi + 20$ **(B)** $10\pi + 30$ **(C)** $10\pi + 40$ **(D)** $20\pi + 20$
(E) $20\pi + 40$

2 3D Shapes

3D shapes surround you every single day! We live in a 3 dimensional world. That's why it's so important to learn 3D geometry. It will also show up on AMC tests, so it's always helpful to know how to work with 3D shapes!

Always remember that 3D shapes will **always** have a volume and surface area! We will be learning some of these formulas later.

2.1 Polyhedrons

Simply defined, a polyhedron is a solid with flat, polygonal faces. Reminder: a polygon is a flat shape with straight sides!

Some common polyhedrons include platonic solids, prisms, and pyramids. If you don't recognize a couple of these, don't worry; you can review quick definitions below!

Platonic Solids are 3D shapes where each face is the same regular polygon. The same number of polygonal faces meet at each vertex. Using this definition, can you write down the platonic solids? *Hint: there are only 5!*

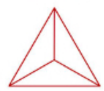
Prisms are solid objects with identical ends, flat faces, and the same cross section all across it's length. We will go into these more later!

Pyramids are solids with a polygonal base and triangular faces. You are likely most familiar with the square pyramid, which has a square base and four triangular faces.

Although it seems like these are all of the 3D shapes, there are more! Which 3D shapes are not polyhedrons? *Hint: look at the original definition of a polyhedron!*

2.1.1 Platonic Solids

There are only five platonic solids! You probably recognize some of them, but you might not see other ones very often. You can see platonic solids and their characteristics below:

Tetrahedron 	• 3 triangles meet at each vertex • 4 Faces • 4 Vertices • 6 Edges
Cube 	• 3 squares meet at each vertex • 6 Faces • 8 Vertices • 12 Edges
Octahedron 	• 4 triangles meet at each vertex • 8 Faces • 6 Vertices • 12 Edges
Dodecahedron 	• 3 pentagons meet at each vertex • 12 Faces • 20 Vertices • 30 Edges
Icosahedron 	• 5 triangles meet at each vertex • 20 Faces • 12 Vertices • 30 Edges

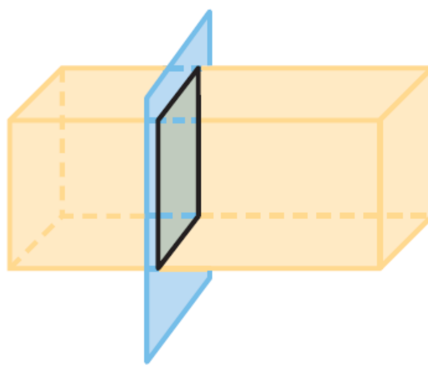
As you can probably tell, you will likely not encounter the last three shapes on any AMC tests. However, you should be familiar with the cube

What is the volume formula for a cube? The surface area?

2.1.2 Prisms

You are probably most familiar with prisms and most likely will encounter them on the AMC tests. Remember that a prism is a solid that has identical ends, flat polygonal faces, and the same cross section through the length.

A cross section is made by cutting an object straight through the length. All cross-sections of the length which are parallel to the bases are the same (and the same as the base). The following image shows the cross section for a rectangular prism.



The cross section of the above prism is a rectangle, thus it is a rectangular prism.

What shape is the cross section of a triangular prism?

Prisms all have 2 parallel bases for which they get their name. Triangular prisms have triangular bases, rectangular prisms have rectangular bases, etc. These bases are always the same as the cross sections. When the base is not a "regular polygon" (a shape with equal side lengths and equal angles), the prism is referred to as an "oblique prism."

Prisms' sides are always parallelograms, most often a rectangle.

Because of these two facts about prisms' sides, and assuming we only care about perpendicular prisms, it is easy to calculate the surface area and volume.

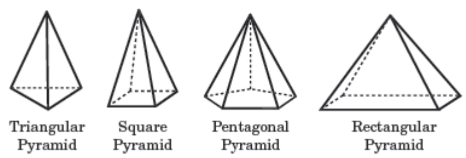
The surface area of a prism is $2A_{base} + P_{base} + l$, where A_{base} is the area of the base, P_{base} is the perimeter of the base, and l is the length of the prism

The volume of a prism can be calculated with $A_{base} \cdot l$, where A_{base} is the area of the base and l is the length of the prism.

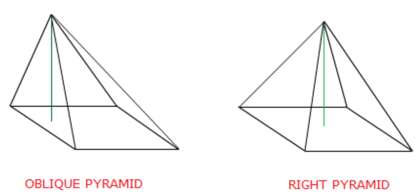
This is why it's so important to know the area for basic shapes, which is why we went over 2D shape formulas last week!

2.1.3 Pyramids

Pyramids have a base and triangular faces. The shape of the base determines the name of the pyramid.



There are different types of pyramids as well! Right pyramids are pyramids where the apex, or vertex where all the triangular faces connect, is directly above the center of the base.



Regular and irregular pyramids are defined by the shape of the base. If the base is a regular polygon, then the pyramid is regular. If the base is not a regular polygon, then the pyramid is irregular.

We can see that it's easy to find the volume and surface of pyramids too!

The volume of a pyramid is always $\frac{1}{3}A_{base} \cdot h$, where A_{base} is the area of the base and h is the heigth of the pyramid.

The surface area of the pyramid differs depending on the side faces.
When all side faces are the same,

$$SA = A_{base} + \frac{1}{2}P_{base} \cdot l_{slant}$$

where A_{base} is the area of the base, P_{base} is the perimeter of the base, and l_{slant} is the length of the slant of the pyramid.

When side faces are different,

$$SA = A_{base} + A_{lateral}$$

where A_{base} is the area of the base and $A_{lateral}$ is the sum of the areas of the other faces.

2.2 Other 3D Shapes

Although there are many polyhedrons, there are some important 3D shapes that are not polyhedrons. These shapes will probably show up on AMC tests, but they are pretty easy to master!

The cylinder has two circular bases and is round! It's your standard soda can shape. For a cylinder with radius r and height h:

$$V = \pi r^2 h$$
$$SA = 2\pi rh + 2\pi r^2$$

A cone is also incredibly important! You've all probably held cones when eating ice cream or wearing a party hat. The cone has a circular base and an apex at the top of a curved side. For a cone with radius r, height h, and slant height s:

$$V = \frac{1}{3}\pi r^2 h$$
$$SA = \pi r^2 + \pi rs$$

Finally, we have a sphere, which is just a ball! For a sphere with radius r:

$$V = \frac{4}{3}\pi r^3$$
$$SA = 4\pi r^2$$

3 Practice

One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

(A) 8 **(B)** 27 **(C)** 64 **(D)** 125 **(E)** 216

Logan is constructing a scaled model of his town. The city's water tower stands 40 meters high, and the top portion is a sphere that holds 100,000 liters of water. Logan's miniature water tower holds 0.1 liters. How tall, in meters, should Logan make his tower?

(A) 0.04 **(B)** $\frac{0.4}{\pi}$ **(C)** 0.4 **(D)** $\frac{4}{\pi}$ **(E)** 4

A truncated cone has horizontal bases with radii 18 and 2. A sphere is tangent to the top, bottom, and lateral surface of the truncated cone. What is the radius of the sphere?

(A) 6 (B) $4\sqrt{5}$ (C) 9 (D) 10 (E) $6\sqrt{3}$

A rectangular box P is inscribed in a sphere of radius r. The surface area of P is 384, and the sum of the lengths of its 12 edges is 112. What is r?

(A) 8 (B) 10 (C) 12 (D) 14 (E) 16

Corners are sliced off a unit cube so that the six faces each become regular octagons. What is the total volume of the removed tetrahedra?

(A) $\frac{5\sqrt{2}-7}{3}$ (B) $\frac{10-7\sqrt{2}}{3}$ (C) $\frac{3-2\sqrt{2}}{3}$ (D) $\frac{8\sqrt{2}-11}{3}$ (E) $\frac{6-4\sqrt{2}}{3}$

What is the volume of a cube whose surface area is twice that of a cube with volume 1?

(A) $\sqrt{2}$ (B) 2 (C) $2\sqrt{2}$ (D) 4 (E) 8

A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top $\frac{1}{8}$ of the volume of the mountain is above water. What is the depth of the ocean at the base of the mountain in feet?

(A) 4000 **(B)** $2000(4-\sqrt{2})$ **(C)** 6000 **(D)** 6400 **(E)** 7000

A pyramid has a square base $ABCD$ and vertex E. The area of square $ABCD$ is 196, and the areas of $\triangle ABE$ and $\triangle CDE$ are 105 and 91, respectively. What is the volume of the pyramid?

(A) 392 **(B)** $196\sqrt{6}$ **(C)** $392\sqrt{2}$ **(D)** $392\sqrt{3}$ **(E)** 784

The circular base of a hemisphere of radius 2 rests on the base of a square pyramid of height 6. The hemisphere is tangent to the other four faces of the pyramid. What is the edge-length of the base of the pyramid?

(A) $3\sqrt{2}$ **(B)** $\frac{13}{3}$ **(C)** $4\sqrt{2}$ **(D)** 6 **(E)** $\frac{13}{2}$

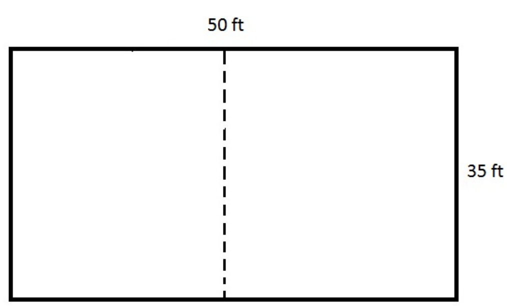
A water tank takes the shape of a sphere whose exterior has radius 16 feet; the tank is three inches thick throughout. To the nearest hundred, give the surface area of the interior of the tank in square feet.

(A)$300ft^2$ **(B)**$16400ft^2$ **(C)**$15600ft^2$ **(D)**$17100ft^2$ **(E)**$3100ft^2$

A rectangular swimming pool is D meters deep throughout and W meters wide. Its length is ten meters greater than twice its width. How many liters of water does the pool hold? (One cubic meter is equal to one thousand liters)

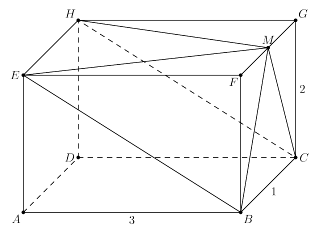
(A)$2,000DW^2{+}10,000DW$ **(B)**$2,000DW^2 10,000DW$ **(C)**$0.002DW^2 0.01DW$
(D)$0.002DW^2 + 0.01DW$ **(E)**None of the above

The below picture depicts a rectangular swimming pool for an apartment. On the left and right edges, the pool is three feet deep; the dashed line at the very center represents the line along which it is eight feet deep. Going from the left to the center, its depth increases uniformly; going from the center to the right, its depth decreases uniformly. In cubic feet, how much water does the pool hold?



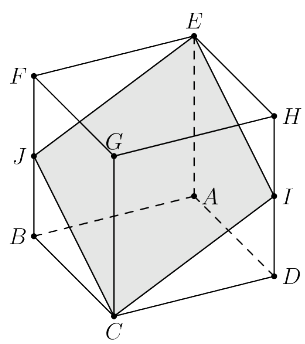
(A)$1,870ft^3$ **(B)**$9,875ft^3$ **(C)**$9,625ft^3$ **(D)**$2,070ft^3$ **(E)**$10,225ft^3$

In the rectangular parallelepiped shown, $AB = 3$, $BC = 1$, and $CG = 2$. Point is the midpoint of . What is the volume of the rectangular pyramid with base $BCHE$ and apex M?



(A) 1 **(B)** $\frac{4}{3}$ **(C)** $\frac{3}{2}$ **(D)** $\frac{5}{3}$ **(E)** 2

In the cube $ABCDEFGH$ with opposite vertices C and E, J and I are the midpoints of segments FB and HD respectively. Let be the ratio of the area of the cross-section $EJCI$ to the area of one of the faces of the cube. What is R^2?



(A) $\frac{5}{4}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{3}{2}$ **(D)** $\frac{25}{16}$ **(E)** $\frac{9}{4}$

Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

(A) $3 + \frac{\sqrt{30}}{2}$ (B) $3 + \frac{\sqrt{69}}{3}$ (C) $3 + \frac{\sqrt{123}}{4}$ (D) $\frac{52}{9}$ (E) $3 + 2\sqrt{2}$

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