An Introduction to Divisibility

Math Circle (Intermediate)

November 4, 2012

1. A few definitions.

   A number is...

   - ________________________ if it is equal to the product of two smaller natural numbers.
   - ________________________ if it is not composite and is not equal to 1.
   - The number ________ is neither prime nor composite.

   Any composite number can be broken down into a product of primes. This product is called its ________________________ or its **prime factorization**. We can thus *factor* any number greater than 1; that is, we can find its decomposition. We just keep factoring the numbers until we are left with only ________________________.

   By the way, the prime factorization of any number is unique. This fact is known as the Fundamental Theorem of Arithmetic. It sounds super important...because it is!

   ## FUNDAMENTAL THEOREM OF ARITHMETIC:

   (The proof of this theorem is actually rather complex, but you can test it with several numbers to convince yourself that it is true.)

2. Test your understanding.

   (a) Is $2^8 \cdot 3$ divisible by 6? How do you know?

   (b) Is $32 \cdot 2^4 \cdot 3$ divisible by 128? How do you know?

3. **Prove or disprove:** If a natural number n is divisible by 4 and by 3, then it must be divisible by $4 \cdot 3 = 12$.

4. **Prove or disprove:** If a natural number n is divisible by 4 and by 6, then it must be divisible by $4 \cdot 6 = 24$.

5. The number A is not divisible by 3.

   (a) Is it possible that the number $2A$ is divisible by 3? Justify your answer.

   (b) How about $3A$? $4A$? Justify.

6. The number 15B is divisible by 6. Is it true that A must also be divisible by 6? Justify your answer.

One more definition:
Two numbers are COPRIME or RELATIVELY PRIME if they have no common divisors greater than 1.
Note that any pair of prime numbers are relative prime, and that 1 is relatively prime to any natural number.

7. Did you notice how similar problems 3 and 4 were? Yet, in one case, the statement was provable, and in the other case, there are infinitely many counterexamples.

   (a) Can you spot the key difference between Problems 3 and 4?

(b) Let's see if we can find another example.

  i. If a natural number n is divisible by 4 and 5, is it also divisible by $4 \cdot 5 = 20$? Prove it or find a counterexample!

  ii. If a natural number n is divisible by 4 and 6, is it also divisible by $4 \cdot 6 = 24$? Prove it or find a counterexample!

(c) Use your results from parts (a) and (b) to form a generalized hypothesis about the divisibility of a number by the product of two smaller numbers.
(Hint: Your hypothesis will have the form "If some natural number is... then it is...")

(d) Prove it!

8. **Prove or disprove:** If the number pA is divisible by q, where p and q are relatively prime, then A is also divisible by q.

9. **Prove or disprove:** The product of any three consecutive natural numbers is divisible by 6. [1]

[1]Some problems are taken from:
D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"