

Big Numbers

1 Introduction

Remember when we were young kids we often ask “What comes after a thousand? After a million? After a billion?”, et cetera. Obviously there is no limit; whatever number you name, a larger one can be obtained simply by adding one. What is interesting is just how large a number can be expressed in a fairly compact way.

Problem 1.1. *Using the standard arithmetic operations (addition, subtraction, multiplication, division and exponentiation), what is the largest number you can make using three copies of the digit “9”?*

In the problem above, you might notice that stacking exponents makes huge numbers very rapidly, and you might at first think that if you’ve got 100 symbols, your best bet would be to stack 100 9s in a tower of exponents and that would pretty much beat all the other possibilities. While this would be a pretty large number, with a little cleverness you can do far better. In this class, we will introduce various smart ways to write a super huge numbers, much larger than you can imagine!

Problem 1.2. Using the standard arithmetic operations (addition, subtraction, multiplication, division and exponentiation), what is the largest number you can make using four copies of the digit “2”?

Problem 1.3. Which number do you think is larger, 2^{400} or 3^{200} ?

2 Scientific Notations

The scientific notation helps us to represent the numbers which are very huge or very tiny in a form of multiplication of single-digit numbers and 10 raised to the power of the respective exponent. The exponent is positive if the number is very large and it is negative if the number is very small. For example, 4500000 can be written into

$$4.5 \times 10^6$$

where 4.5 is called the *coefficient*, and 6 is called the *exponent*.

The diagram illustrates the conversion of two numbers into scientific notation. The first line shows $4500000 = 4.5 \times 10^6$. The second line shows $0.00453 = 4.53 \times 10^{-3}$. A bracket on the right side groups the 10^6 and 10^{-3} terms, with the label "Exponent" pointing to it. A vertical line points from the label "Coefficient" to the 4.5 in the first equation and the 4.53 in the second equation. Another vertical line points from the label "Base" to the 10 in both equations.

More generally, a scientific notation of a nonzero number is in the form of

$$a \times 10^b$$

where $1 \leq a < 10$ is written in decimals and b is an integer.

Problem 2.1. Write the following numbers into correct scientific notations:

- 20000000 =
- 56209000000000 =
- 0.0000032 =
- $52.4 \times 10^8 =$
- $0.032 \times 10^{12} =$

Problem 2.2. Write the following results into scientific notations:

- $5 \times 10^6 + 3 \times 10^5 =$
- $4.2 \times 10^6 - 2 \times 10^8 =$
- $(2 \times 10^{20}) \times (5.5 \times 10^{-2}) =$
- $(5 \times 10^6)/(3 \times 10^8) =$
- $2^{2^{2^2}} =$

Problem 2.3. *The speed of light is approximately 3×10^8 meters per seconds, while the distance to go around the Earth once is about 4×10^7 meters. How many loops can light travel around the Earth in 1 second?*

How are the big numbers named? The common names for numbers that we may be familiar with are:

Number	Scientific Notation	American Name	English Name
1,000	10^3	Thousand	Thousand
1,000,000	10^6	Million	Million
1,000,000,000	10^9	Billion	Thousand Million
1,000,000,000,000	10^{12}	Trillion	Billion
1,000,000,000,000,000	10^{15}	Quadrillion	Thousand Billion
1,000,000,000,000,000,000	10^{18}	Quintillion	Trillion

The names continue in the same general way, and after Quintillion, they are: Sextillion (10^{21}), Septillion (10^{24}), Octillion (10^{27}), Nonillion (10^{30}), Decillion (10^{33}), Undecillion (10^{36}), and so on. The numbers in parentheses above correspond to the American names. To obtain the values in the English naming system, subtract 3 from each exponent and then double it.

The name Googol stands for 10^{100} - a 1 followed by a hundred zeroes; and the name Googolplex means the number 1 followed by a Googol zeroes.

Problem 2.4. Write down the number Googolplex in the form of stacking exponents.

Problem 2.5. The speed of light is approximately 3×10^8 meters per seconds. How many years does it take for light to travel a Googol meters? (Ask the instructors for calculators if needed.)

Problem 2.6. The thickness of a paper is about 0.1 millimeter. When you fold a piece of paper, its thickness doubles. The distance from the Earth to the Sun is 149 million kilometers. Which distance is bigger, the thickness of a paper folded 60 times or the distance from the Earth to the Sun?

3 Even Bigger Numbers

Recall the factorial of a natural number is defined as

$$n! = n \times (n - 1) \times \cdots \times 1.$$

With factorials, we can get some super big numbers.

Problem 3.1. *For each of the following pairs, compare which number is bigger. Explain.*

- $10!$ and 3^{10} .

- $100!$ and a Googol.

- A Googolplex and $(\text{Googol})!$.

Factorials can give use big number, but it turns out that the following method gives even bigger numbers in very simple notations.

First, we begin by saying what we mean by \triangle_n (which we will pronounce as “triangle-n”). It will simply be defined as n^n :

$$\triangle_n = n^n.$$

Next, define “square- n ” \boxed{n} to be \triangle surrounded by n triangle. For example, square-2 is the same as triangle-triangle-2, which is triangle- $2^2 = \text{triangle-}4$, which is $4^4 = 256$.

Finally, define “pentagon- n ” \hat{n} to be “square- n ” surrounded by n square.

For example, “pentagon-2” is the same as “square-square-2”, which we know is “square-256”. Now how big is “square-256”? It’s the number 256 surrounded by 256 triangles. To get rid of the innermost triangle, we obtain 256^{256} , a number with 616 digits, surrounded by 255 triangles. Raising this 616-digit number to its own power will leave us with “only” 254 triangles, et cetera.

Problem 3.2. Write “square-3” in terms of stacking exponents.

Problem 3.3. What is “pentagon-1”?

Problem 3.4. Which is bigger, “pentagon-2” or a Googolplex?

Problem 3.5. *In similar way, define “hexagon- n ” and “heptagon- n ”.*

Problem 3.6. *Define a general polygon notation iteratively.*

Problem 3.7. *The Moser number is defined as 2 surrounded by a polygon with \diamondsuit^2 sides. Is the Moser number bigger than a Googolplex to the Googolplex’s power? Explain.*

Problem 3.8. *Design your own mathematical symbol, such that you can write down a number bigger than the Moser number within at most 5 digits, symbols, and letters.*

4 Ackermann's Function

Recall that a (numerical) function takes in one or more numbers as input and output another number. For example, the function $f(x) = x^2 - 3$ will take the number $x = 1$ to the number $1^2 - 3 = -2$, and take the number $x = 2$ to the number $2^2 - 3 = 1$, and so on.

A function can have multiple inputs. For example, the function $f(x, y) = x + y$ takes in two numbers, and output the sum of the two numbers.

One special type of function is called recursive function. For example, the function

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + F(n - 1) & \text{if } n > 1 \end{cases}$$

is defined over all positive integers. From the definition, we have

$$F(1) = 1$$

$$F(2) = 2 + F(1) = 2 + 1 = 3$$

$$F(3) = 3 + F(2) = 3 + 3 = 6$$

and so on.

Problem 4.1. *Use the same function F as above. What is the value of $F(4)$, $F(5)$, $F(6)$? What about $F(100)$?*

Problem 4.2. Define the function

$$G(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ n + F(n - 1) + F(n - 2) & \text{if } n > 2 \end{cases}$$

Find $G(10)$.

Problem 4.3. Write the definition of factorial in terms of an iterative function.

Now let's define the Ackermann's Function.

Definition 1. The Ackermann's Function is defined for any natural numbers m, n by the following:

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m - 1, 1) & \text{if } m > 0, n = 0 \\ A(m - 1, A(m, n - 1)) & \text{if } m > 0, n > 0 \end{cases}$$

Problem 4.4. Fill out the following table with values of the Ackermann's Function. Each row represents a value of m and each column represents a value of n .

$m \setminus n$	0	1	2	3	4	5	6	7	8
0	1	2	3	4	5				
1	2	3							
2									
3									
4									

Problem 4.5. Show that

$$A(4, n) = 2^{2^{\dots^2}} - 3$$

where there are $n + 3$ copies of 2.

Problem 4.6. Describe how large is $A(5, 1)$ using stacking exponents. (Note: it's impossible to write this number out. Just describe how high it is to write such a number in a power tower of 2).

Problem 4.7. (Challenge) Let A, B, C, D, E be five variables, initially zero. Bob designed a new language, which has only three types of statements:

- $A++$ means incrementing the value of A by 1, then go to the next line;
- $A--$ means decrementing the value of A by 1 unless $A = 0$, then go to the next line;
- $A :: x$ means if $A \neq 0$, then go to the line numbered x ; otherwise do nothing and go to the next line.

In this new language, Bob then wrote 23 lines of instructions:

1: $E++$	9: $A++$	17: $D--$
2: $E++$	10: $B--$	18: $D :: 5$
3: $E++$	11: $B :: 8$	19: $D++$
4: $E++$	12: $C--$	20: $C--$
5: $B++$	13: $C :: 5$	21: $C :: 19$
6: $A--$	14: $C++$	22: $E--$
7: $A :: 5$	15: $A--$	23: $E :: 5$
8: $A++$	16: $A :: 14$	

Alice then read this starting from line 1 and follow the instructions one by one. She will stop if she reaches line 23 and the statement in line 23 tells her to go to the next line. At the time she stops, how large do you think A is?