Complex Numbers in Geometry

ORMC

02/12/24

1 Book Problems: Problem Solving Through Problems

Problem 1.1 (Proble)**.** Use complex numbers to show that if a, b, and n are positive integers, prove there exist integers x and y such that

$$(a^2 + b^2)^n = x^2 + y^2.$$

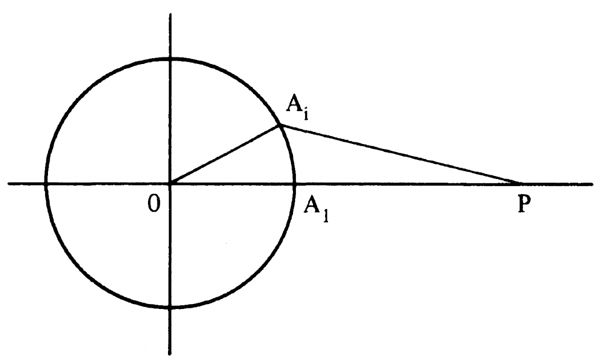
Problem 1.2. Let n be an integer > 3, and let α, β, γ be complex numbers such that $\alpha^n = \beta^n = \gamma^n = 1$, $\alpha + \beta + \gamma = 0$. Show that n is a multiple of 3.

Hint: Where do these numbers lie in the plane? If $\alpha = 1$, what can you learn about β and γ?

Problem 1.3. $A_1, A_2, \ldots, A_n$ are vertices of a regular polygon inscribed in a circle of radius r and center O. P is a point on OA_1 extended beyond A_1. Show that

$$\prod_{k=1}^{n} |PA_k| = |OP|^n - r^n.$$

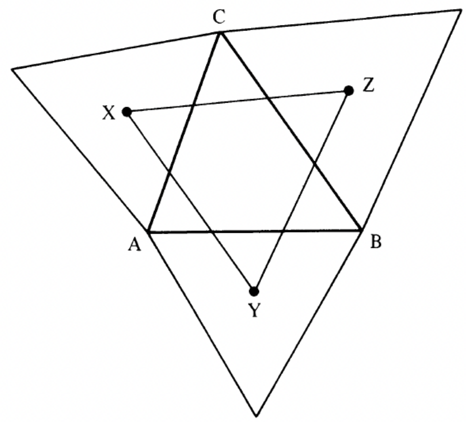
Problem 1.4. Given a point P in the complex plane and the vertices $A_1, A_2, \ldots, A_n$ of a regular polygon of n sides inscribed in a circle, prove that $|PA_1|^2 + |PA_2|^2 + \cdots + |PA_n|^2$ depends only on n and the distance between P and the center of the circle.



Problem 1.5. If z_1, z_2 are distinct points in the complex plane, and are two vertices of an equilateral triangle, find formulas for the two possible third vertices.

Hint: If z_1, z_2, z_3 are the vertices of an equilateral triangle, we know the angle between the vertices. Can you relate $z_2 - z_1$ to $z_3 - z_2$?

Problem 1.6. Equilateral triangles are erected externally on the sides of an arbitrary triangle ABC. Prove that the centers (centroids) of these three equilateral triangles form an equilateral triangle.



2 Putnam Problems

Problem 2.1 (2008 Putnam A5)**.** Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

Problem 2.2 (2018 Putnam B2)**.** Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

3 USAMO Problem, hints based on Evan Chen's Notes

This section builds up to a problem using linear algebra ideas. I've hopefully given you enough background to solve it even if you haven't done much of this, but please ask me some questions about this determinant function.

Problem 3.1. If A is a point in the complex plane, give a formula for the reflection of A over the real line.

Problem 3.2. Suppose A, B, C, D are points in the complex plane. What does the complex number $\frac{D-C}{B-A}$ tell us about the line segments AB and CD?

Problem 3.3. How can you test if three points A, B, C are collinear? (Hint: What about the case where $A = 0$?)

Problem 3.4. Show that the intersection point between AB and $C0$ can be given by

$$\frac{(\bar{a}b - a\bar{b})c}{(\bar{a} - \bar{b})c - (a-b)\bar{c}}.$$

Hint: Use your collinearity test.

Problem 3.5. Let P be a point in the plane of ΔABC, and γ a line through P. Let A' be the point where the reflection of the line PA with respect to γ intersects the line BC.

Assuming that $P = 0$ and γ is the real line, find a complex expression for A'.

Problem 3.6. The *determinant* of a 3 by 3 matrix is the sum of the products of the three increasing diagonals minus the sum of the products of the three decreasing diagonals:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Here are some facts about determinants:

- Multiplying a row or column by a number multiplies the determinant by that number.
- If one row is the sum of two other rows, then the determinant is 0.

Optionally, show that the points A, B, C in the complex plane are collinear if this equation is true:

$$\det \begin{bmatrix} A & B & C \\ \bar{A} & \bar{B} & \bar{C} \\ 1 & 1 & 1 \end{bmatrix} = 0.$$

Hint: Show that you can multiply by a complex number to rotate the line through A, B, C to be vertical (perpendicular to the real line). Multiply the rows by carefully chosen numbers and add to show that the determinant is 0.

Problem 3.7 (USAMO 2012)**.** Let P be a point in the plane of ΔABC, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.