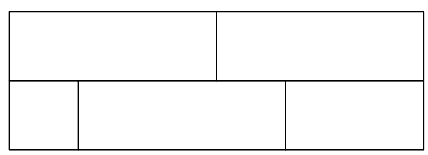
ORMC AMC 10/12 Group
Week 7: Combinatorics

February 18, 2023

1 Warm-up Exercises

1. **(2018 AMC 12B #5)** How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

2. **(2022 AMC 10A #9)** A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



3. **(2018 AMC 12B #15)** How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

4. **(2017 AMC 12B #16)** The number $21! = 51{,}090{,}942{,}171{,}709{,}440{,}000$ has over $60{,}000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

5. **(2017 AMC 10B #17)** Call a positive integer monotonous if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

2 Theorems/Techniques

2.1 Stars-and-Bars

gives a way to count how *indistinguishable items* can be sorted into *distinguishable containers*. If the problem you are working on can be translated into the form "How many ways can n identical balls be placed into m labeled boxes?", then you can use stars-and-bars.

Write out n stars ($*$), and draw $m-1$ bars ($|$), separating the stars. The different boxes are indicated by the spaces between the bars. For example:

$$* * * * || \implies \text{4 balls in box 1, 0 balls in box 2, 0 balls in box 3}$$

$$* * *| * | \implies \text{3 balls in box 1, 1 ball in box 2, 0 balls in box 3}$$

$$*| * *|* \implies \text{1 ball in box 1, 2 balls in box 2, 1 ball in box 3}$$

The total number of star-and-bar layouts is given by finding how many ways there are to place the n stars among the total spots for n stars $+\ m-1$ bars. This value is $\boxed{\binom{n+m-1}{n}}$.

2.2 Burnside's Lemma

Burnside's Lemma is useful for counting problems where symmetries (rotation/reflection) should be ignored.

Let S be the set of all configurations before ignoring symmetries, and let G be the set of all symmetries/transformations. Burnside's lemma tells us that the total number of distinct configurations, *after eliminating symmetries*, is:

$$\frac{1}{|G|}\sum_{g\in G}|\text{fix}(g)| = \frac{1}{|G|}\sum_{c\in S}|\text{stab}(c)| = \frac{1}{|G|}\left|\{(g,c) \mid g(c)=c\}\right|$$

Where $\text{stab}(c)$ is the *stabilizer* of c, the set of transformations that do not change c; $\text{fix}(g)$ is the fixed points of g, the configurations that are not changed by g. The set on the right-hand side contains all pairs of transformations and configurations where the configuration remains unchanged by the transformation.

For a regular n-gon, there are n possible rotations (including the $0°$ rotation), and n possible axes of reflection, for a total of $2n$ possible transformations.

2.2.1 Example:

We can color the vertices of a square red or blue. Two colorings which can be obtained from one another by a rotation or a reflection should be considered the same. How many different colorings are there?

Solution: Without worrying about rotations or reflections, there are $2^4 = 16$ different configurations. In order to apply Burnside's Lemma, we'll count $|\text{fix}(g)|$ for the 8 transformations:

1. The identity transformation (no rotation/reflection) fixes all 16 configurations
2. The $90°$ and $270°$ rotations only fix the all-red and all-blue configurations.
3. the $180°$ rotation only fixes configurations where diagonal vertices have the same color. There are $2^2 = 4$ of these.
4. The vertical and horizontal reflections allow us to choose colors for 2 vertices, so there are $2^2 = 4$ possible combinations
5. The diagonal reflections allow us to choose colors for 3 vertices, so there are $2^3 = 8$ possible configurations.

Putting this all together, the number of possible combinations is:

$$\frac{1}{8}(16 + 2\cdot 2 + 4 + 2\cdot 4 + 2\cdot 8) = \boxed{6}.$$

There are often quicker methods which use casework or intuition, but Burnside's Lemma gives very methodical and dependable solutions, which can often be good to have as a backup.

3 Exercises

1. **(2019 AMC 8 #25)** Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

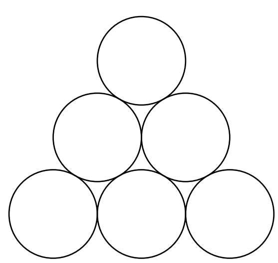
2. Henry's little brother has 8 identical stickers and 4 sheets of paper, each a different color. He puts all the stickers on the pieces of paper. How many ways are there for him to do this, if only the number of stickers on each sheet of paper matters?

3. **(2013 AMC 12A #15)** Rabbits Peter and Pauline have three offspring—Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

4. **(Mock AIME 3 Pre 2005 #2)** Let N denote the number of 7 digit positive integers have the property that their digits are in increasing order. Determine the remainder obtained when N is divided by 1000. (Repeated digits are allowed.)

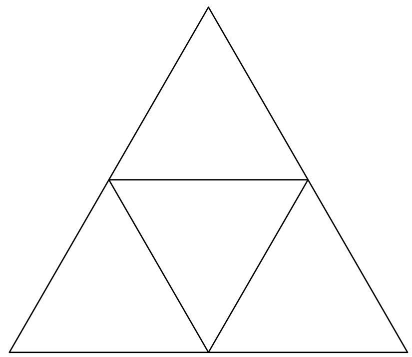
5. **(2014 AMC 12A #13)** A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

6. **(2017 AMC 12B #13)** In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?



7. **(2006 AIME II #8)** There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small triangles are of the same color.

   Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles may be formed?



8. **(2018 AMC 10A #20)** A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

9. **(2012 AMC 10A #20)** A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated $90\,^\circ$ clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black?

10. **(2021 AMC 12A #23)** Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop-up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she ”wraps around” and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops ”up”, the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

11. Matt's four cousins are coming to visit. There are four identical rooms that they can stay in. If any number of the cousins can stay in one room, how many different ways are there to put the cousins in the rooms?

12. **(2016 AMC 10A #18)** Each vertex of a cube is to be labeled with an integer 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?

13. **(2012 AMC 10A #23)** Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

14. **(2020 AMC 10B #25)** Let $D(n)$ denote the number of ways of writing the positive integer n as a product
$$n = f_1 \cdot f_2 \cdots f_k,$$
where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

15. Six indistinguishable balls are to be distributed into three indistinguishable boxes. How many ways are there to do this?

16. **(1986 AIME #13)** In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence TTTHHTHTTTHHTTH of 15 coin tosses we observe that there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?

17. How many ways are there to put 9 differently colored beads on a 3×3 grid if the purple bead and the green bead cannot be adjacent (either horizontally, vertically, or diagonally), and rotations and reflections of the grid are considered the same?

18. **(2010 AMC 12A #18)** A 16-step path is to go from $(-4,-4)$ to $(4,4)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \le x \le 2$, $-2 \le y \le 2$ at each step?

19. **(2021 Fall AMC 12B #20)** A cube is constructed from 4 white unit cubes and 4 blue unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

20. **(2018 AMC 12B #22)** Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0,1,2,3,4,5,6,7,8,9\}$. How many such polynomials satisfy $P(-1) = -9$?