Nonstandard Analysis

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Based on handouts by Nikita and Stepan

Part 1: Nonarchimedian Extensions

Definition 1:
An *ordered field* consists of a set S, the operations $+$ and $\times$, and the relation $<$.
An ordered field must satisfy the following properties:

- **Properties of $+$:**
  - Commutativity: $a + b = b + a$
  - Associativity: $a + (b + c) = (a + b) + c$
  - Identity: there exists an element 0 so that $a + 0 = a \forall a \in S$
  - Inverse: for every $-a$, there exists a $-a$ so that $a + (-a) = 0$
- **Properties of $\times$:**
  - Commutativity
  - Associativity
  - Identity (which we label 1)
  - Inverse (which we label a^{-1}, and which doesn't exist for 0)
  - Distributivity: $a(b + c) = ab + ac$
- **Properties of $<$:**
  - Non-reflexive: $x < x$ is always false
  - Transitive: $x < y$ and $y < z$ imply $x < z$
  - Connected: for all $x, y \in S$, either $x < y$, $y > x$, or $x = y$.
  - If $x < y$ then $x + z < y + z$
  - If $x < y$ and $z > 0$, then $xz < yz$
  - $0 < 1$

Definition 2:
An ordered field that contains $\mathbb{R}$ is called a *nonarchimedian extension* of $\mathbb{R}$.

Problem 3:
Show that each of the following is true in any ordered field.
A: if $x \neq 0$ then $(x^{-1})^{-1} = x$
B: $0 \times x = x$
C: $(-x)(-y) = xy$
D: if $0 < x < y$, then $x^{-1} > y^{-1}$

Definition 4:
In an ordered field, the *magnitude* of a number x is defined as follows:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

Definition 5:
We say an element δ of an ordered field is *infinitesimal* if $|n\delta| < 1$ for all $n \in \mathbb{Z}^+$.
Note that $\mathbb{Z}^+$ is a subset of any nonarchimedean extension.

Likewise, we say x is *limited* if $|x| < n$ for some $n \in \mathbb{Z}^+$.
Elements that are not limited are *unlimited*.

Definition 6:
We say an element x of a field is *positive* if $x > 0$.
We say x is *negative* if $x < 0$.

Problem 7:
Show that a positive δ is infinitesimal if and only if $\delta < x$ for all $x \in \mathbb{R}^+$.
Then, show that a negative δ is infinitesimal if and only if it is bigger than every $x \in \mathbb{R}^-$.

Problem 8:
Prove the following statements:

- If δ and ε are infinitesimal, then $\delta + \varepsilon$ is infinitesimal.
- If δ is infinitesimal and x is limited, then $a\delta$ is infinitesimal.
- If x and y are limited, xy and $x + y$ are too.
- A nonzero δ is infinitesimal iff δ^{-1} is unlimited.

Problem 9:
Let δ be a positive infinitesimal. Which is greater?

- δ or δ^2?
- $(1 - \delta)$ or $(1 + \delta^2)^{-1!}$?
- $\frac{1+\delta}{1+\delta^2}$ or $\frac{2+\delta^2}{2+\delta^3}$?
  Note: we define $\frac{1}{x}$ as x^{-1}, and thus $\frac{a}{b} = a \times b^{-1}$

Definition 10:
We say two elements of an ordered field are *infinitely close* if $x - y$ is infinitesimal.
We say that $x_0 \in \mathbb{R}$ is a *standard part* of x if it is infinitely close to x.

Problem 11:
We will denote the standard part of x as $\mathrm{st}(x)$.
Show that $\mathrm{st}(x)$ is well-defined for limited x.
(In other words, Show that x_0 exists and is unique for limited x).
Hint: To prove existance, consider $\sup(\{a \in \mathbb{R} \mid a < x\})$

Problem 12:
Show that $\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y)$ and $\mathrm{st}(xy) = \mathrm{st}(x)\mathrm{st}(y)$.

Part 2: Dual Numbers

Definition 13:
In the problems below, ε an infinitesimal so that $\varepsilon^2 = 0$.
Note that $\varepsilon \neq 0$.

Definition 14:
The set of *dual numbers* is a nonarchimedian extension of $\mathbb{R}$ that consists of elements that look like $a + b\varepsilon$, where $a, b \in \mathbb{R}$.

Problem 15:
Compute $(a + b\varepsilon) \times (c + d\varepsilon)$.

Definition 16:
Let $f(x)$ be an algebraic function $\mathbb{R} \to \mathbb{R}$.
(that is, a function we can write using the operators $+ - \times \div$, powers, and roots)
Note: Why this condition? These are the only operations we have in an ordered field!
Powers, roots, and division aren't directly available, but are fairly easy to define.

the *derivative* of such an f is a function f' that satisfies the following:

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon$$

If $f(x + \varepsilon)$ is not defined, we will say that f is not *differentiable* at x.

Problem 17:
What is the derivative of $f(x) = x^2$?

Problem 18:
What is the derivative of $f(x) = x^n$?

Problem 19:
Say the derivatives of f and g are known.
Find the derivatives of $h(x) = f(x) + g(x)$ and $k(x) = f(x) \times g(x)$.

Problem 20:
When can you divide dual numbers?
That is, for what numbers $(a + b\varepsilon)$ is there a $(x + y\varepsilon)$ such that $(a + b\varepsilon)(x + y\varepsilon) = 1$?

Problem 21:
Find an explicit formula for the inverse of a dual number,
and use it to find the derivative of $f(x) = \frac{1}{x}$.

Problem 22:
Which dual numbers have a square root?
That is, for which dual numbers $(a + b\varepsilon)$ is there a dual number $(x + y\varepsilon)$ such that $(x + y\varepsilon)^2 = a + b\varepsilon$?

Problem 23:
Find an explicit formula for the square root and use it to find the derivative of $f(x) = \sqrt{x}$

Problem 24:
Find the derivative of the following functions:
- $f(x) = \frac{x}{1+x^2}$
- $g(x) = \sqrt{1 - x^2}$

Problem 25:
Say the derivatives of f and g are known.
What is the derivative of $f(g(x))$?

Bonus: The supremum & infimum

Definition 26:
In this section, we'll define a "real number" as a decimal, infinite or finite.

Problem 27:
Write 2.317171717... as a simple fraction.

Problem 28:
Write $^2/_{11}$ as an infinite decimal and prove that your answer is correct.

Problem 29:
Show that $0.999... = 1$
Note: There is no real number 0.0...1 with a digit 1 "at infinity."
Some numbers have two decimal representations, some have only one.

Problem 30:
Concatenate all the natural numbers in order to form 0.12345678910111213....
Show that the resulting decimal is irrational.

Problem 31:
Show that a rational number exists between any two real numbers.

Definition 32:
Let M be a subset of $\mathbb{R}$.
We say $c \in \mathbb{R}$ is an *upper bound* of M if $c \geq m$ for all $m \in M$.
The smallest such c is called the *supremum* of M, and is denoted $\sup(M)$.

Similarly, $x \in \mathbb{R}$ is a *lower bound* of M if $x \leq m \forall m \in M$.
The largest upper bound of M is called the *infimum* of M, denoted $\inf(M)$.

Problem 33:
Show that x is the supremum of M if and only if...

- For all $m \in M$, $m \leq x$
- For any $x_0 < x$, there exists an $m \in M$ so that $m > x_0$

Problem 34:
Show that any subset of $\mathbb{R}$ has at most one supremum and at most one infimum.

Problem 35:
Find the supremum and infimum of the following sets:

- $\left\{a^2 + 2a \middle| -5 < a < 5\right\}$
- $\left\{\pm\frac{n}{2n+1} \middle| n \in \mathbb{N}\right\}$

Problem 36:
Let A and B be subsets of $\mathbb{R}$, and let $\sup(A)$ and $\sup(B)$ be known.

- $\sup(A \cup B)$
- $\sup(A + B)$, where $A + B = a + b \forall (a, b) \in A \times B$,
- $\inf(A \cdot B)$, where $A \cdot B = ab \forall (a, b) \in A \times B$

Theorem 37: Completeness Axiom
Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Problem 38:
Show that $a < \sup(A)$ if and only if there is a c in A where $a < c$

Problem 39:
Use the definitions in this handout to prove Theorem 37.
Hint: Build the supremum one digit at a time.

Problem 40:
Let $[a_1, b_1] \subseteq [a_2, b_3] \subseteq [a_3, b_3] \subseteq ...$ be an infinite sequence of closed line intervals. Show that there exists a $c \in \mathbb{R}$ that lies in all of them.
Is this true for open intervals?

Problem 41: Bonus
Show that every real number in $[0, 1]$ can be written as a sum of 9 numbers
Whose decimal representations only contain 0 and 8.

Problem 42: Bonus
Two genies take an infinite amount of turns and write the digits of an infinite decimal. The first genie, on every turn, writes any finite amount of digits to the tail of the decimal. The second genie writes one digit to the end. If the resulting decimal after an infinite amount of turns is periodic, the first genie wins; otherwise, the second genie wins. Who has a winning strategy?