Nonstandard Analysis

Prepared by Mark on April 3, 2024
Based on handouts by Nikita and Stepan

The supremum & the infimum

Definition 1:
In this section, we'll define a "real number" as a decimal, infinite or finite.

Problem 2:
Write 2.317171717... as a simple fraction.

Problem 3:
Write $2/11$ as an infinite decimal and prove that your answer is correct.

Problem 4:
Show that $0.999... = 1$
Note: There is no real number 0.0...1 with a digit 1 "at infinity."
Some numbers have two decimal representations, some have only one.

Problem 5:
Concatenate all the natural numbers in order to form 0.12345678910111213....
Show that the resulting decimal is irrational.

Problem 6:
Show that a rational number exists between any two real numbers.

Definition 7:
Let M be a subset of $\mathbb{R}$.
We say $c \in \mathbb{R}$ is an *upper bound* of M if $c \geq m$ for all $m \in M$.
The smallest such c is called the *supremum* of M, and is denoted $\sup(M)$.

Similarly, $x \in \mathbb{R}$ is a *lower bound* of M if $x \leq m$ for all $m \in M$.
The largest upper bound of M is called the *infimum* of M, denoted $\inf(M)$.

Problem 8:
Show that x is the supremum of M if and only if...

- for all $m \in M$, $m \leq x$, and
- for any $x_0 < x$, there exists an $m \in M$ so that $m > x_0$

Problem 9:
Show that any subset of $\mathbb{R}$ has at most one supremum and at most one infimum.

Problem 10:
Find the supremum and infimum of the following sets:

- $\left\{a^2 + 2a \middle| -5 < a < 5\right\}$
- $\left\{\pm\frac{n}{2n+1} \middle| n \in \mathbb{N}\right\}$

Problem 11:
Let A and B be subsets of $\mathbb{R}$, and let $\sup(A)$ and $\sup(B)$ be known.
Compute the following in terms of $\sup(A)$ and $\sup(B)$.

- $\sup(A \cup B)$
- $\sup(A + B)$, where $A + B = a + b \forall (a, b) \in A \times B$,
- $\inf(A \cdot B)$, where $A \cdot B = ab \forall (a, b) \in A \times B$

Problem 12:
Prove the assumptions in Definition 44:
Show that $\text{st}(x)$ is exists and is unique for limited x.

Theorem 13: Completeness Axiom
Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Problem 14:
Show that $a < \sup(A)$ if and only if there is a c in A where $a < c$

Problem 15:
Use the definitions in this handout to prove Theorem 13.
Hint: Build the supremum one digit at a time.

Problem 16:
Let $[a_1, b_1] \supseteq [a_2, b_3] \supseteq [a_3, b_3] \supseteq ...$ be an infinite sequence of closed line intervals.
Show that there exists a $c \in \mathbb{R}$ that lies in all of them. Is this true of open intervals?

Problem 17: Bonus
Show that every real number in $[0, 1]$ can be written as a sum of 9 numbers
Whose decimal representations only contain 0 and 8.

Problem 18: Bonus
Two genies take an infinite amount of turns and write the digits of an infinite decimal. The first genie, on every turn, writes any finite amount of digits to the tail of the decimal. The second genie writes one digit to the end. If the resulting decimal after an infinite amount of turns is periodic, the first genie wins; otherwise, the second genie wins. Who has a winning strategy?

Part 1: Dual Numbers

Definition 19:
In the problems below, let ε a positive infinitesimal so that $\varepsilon^2 = 0$.
Note that $\varepsilon \neq 0$.

Definition 20:
The set of *dual numbers* consists of elements of the form $a + b\varepsilon$, where $a, b \in \mathbb{R}$.

Problem 21:
Compute $(a + b\varepsilon) \times (c + d\varepsilon)$.

Definition 22:
Let $f(x)$ be an algebraic function $\mathbb{R} \to \mathbb{R}$.
(that is, a function we can write using the operators $+ - \times \div$ and integer powers)

the *derivative* of such an f is a function f' that satisfies the following:

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon$$

If $f(x + \varepsilon)$ is not defined, we will say that f is not *differentiable* at x.

Problem 23:
What is the derivative of $f(x) = x^2$?

Problem 24:
What is the derivative of $f(x) = x^n$?

Problem 25:
Assume that the derivatives of f and g are known.
Find the derivatives of $h(x) = f(x) + g(x)$ and $k(x) = f(x) \times g(x)$.

Problem 26:
When can you divide dual numbers?
That is, for what numbers $(a + b\varepsilon)$ is there a $(x + y\varepsilon)$ such that $(a + b\varepsilon)(x + y\varepsilon) = 1$?

Problem 27:
Find an explicit formula for the inverse of a dual number $(a + b\varepsilon)$, assuming one exists.
Then, use this find the derivative of $f(x) = \frac{1}{x}$.

Problem 28:
Which dual numbers have a square root?
That is, for which dual numbers $(a + b\varepsilon)$ is there a dual number $(x + y\varepsilon)$ such that $(x + y\varepsilon)^2 = a + b\varepsilon$?

Problem 29:
Find an explicit formula for the square root and use it to find the derivative of $f(x) = \sqrt{x}$

Problem 30:
Find the derivative of the following functions:
- $f(x) = \frac{x}{1+x^2}$
- $g(x) = \sqrt{1 - x^2}$

Problem 31:
Assume that the derivatives of f and g are known.
What is the derivative of $f(g(x))$?

Part 2: Extensions of $\mathbb{R}$

Definition 32:
An *ordered field* consists of a set S, the operations $+$ and $\times$, and the relation $<$.
An ordered field must satisfy the following properties:

- **Properties of $+$:**
  - Commutativity: $a + b = b + a$
  - Associativity: $a + (b + c) = (a + b) + c$
  - Identity: there exists an element 0 so that $a + 0 = a$ for all $a \in S$
  - Inverse: for every a, there exists a $-a$ so that $a + (-a) = 0$
- **Properties of $\times$:**
  - Commutativity
  - Associativity
  - Identity (which we label 1)
  - For every $a \neq 0$, there exists an inverse a^{-1} so that $aa^{-1} = 1$
  - Distributivity: $a(b + c) = ab + ac$
- **Properties of $<$:**
  - Non-reflexive: $x < x$ is always false
  - Transitive: $x < y$ and $y < z$ imply $x < z$
  - Connected: for all $x, y \in S$, either $x < y$, $x > y$, or $x = y$.
  - If $x < y$ then $x + z < y + z$
  - If $x < y$ and $z > 0$, then $xz < yz$
  - $0 < 1$

Definition 33:
An ordered field that contains $\mathbb{R}$ is called an *extension* of $\mathbb{R}$.

Definition 34:
The *Archimedian property* states the following:
For all positive x, y, there exists an n so that $nx \geq y$.

Theorem 35:
All extensions of $\mathbb{R}$ are nonarchemedian.
Proving this is difficult.

Problem 36:
Which of the following are ordered fields?

- $\mathbb{R}$ with the usual definitions of $+$, $\times$, $<$
- $\mathbb{R}$ with the usual definitions of $+$, $\times$, $\leq$
  Note that our relation here is $\leq$, not $<$
- $\mathbb{Z}$ with the usual definitions of $+$, $\times$, $<$
- $\mathbb{Q}$ with the usual definitions of $+$, $\times$, $<$
- $\mathbb{C}$ with the usual definitions of $+$, $\times$,
  and with $(a + bi) < (c + di)$ iff $a < c$.

Problem 37:
Show that each of the following is true in any ordered field.
The list of field axioms is provided below, for convenience.
A: if $x \neq 0$ then $(x^{-1})^{-1} = x$
B: $0 \times x = 0$
C: $(-x)(-y) = xy$
D: if $0 < x < y$, then $x^{-1} > y^{-1}$

- **Properties of $+$:**
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- **Properties of $\times$:**
  - Commutativity
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  - Distributivity: $a(b + c) = ab + ac$
- **Properties of $<$:**
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  - Transitive: $x < y$ and $y < z$ imply $x < z$
  - Connected: for all $x, y \in S$, either $x < y$, $x > y$, or $x = y$.
  - If $x < y$ then $x + z < y + z$
  - If $x < y$ and $z > 0$, then $xz < yz$
  - $0 < 1$

Definition 38:
In an ordered field, the *magnitude* of a number x is defined as follows:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

Definition 39:
We say an element δ of an ordered field is *infinitesimal* if $|nd| < 1$ for all $n \in \mathbb{Z}^+$.
Note that $\mathbb{Z}^+$ is a subset of any nonarchimedian extension of $\mathbb{R}$.

Likewise, we say x is *limited* if $|x| < n$ for some $n \in \mathbb{Z}^+$.
Elements that are not limited are *unlimited*.

Definition 40:
We say an element x of a field is *positive* if $x > 0$.
We say x is *negative* if $x < 0$.

Problem 41:
Show that a positive δ is infinitesimal if and only if $\delta < x$ for all $x \in \mathbb{R}^+$.
Then, show that a negative δ is infinitesimal if and only if $\delta > x$ for every $x \in \mathbb{R}^-$.

Problem 42:
Prove the following statements:

- If δ and ε are infinitesimal, then $\delta + \varepsilon$ is infinitesimal.
- If δ is infinitesimal and x is limited, then $x\delta$ is infinitesimal.
- If x and y are limited, xy and $x + y$ are too.
- A nonzero δ is infinitesimal iff δ^{-1} is unlimited.

Problem 43:
Let δ be a positive infinitesimal. Which is greater?

- δ or δ^2
- $(1 - \delta)$ or $(1 + \delta^2)^{-1}$
- $\frac{1+\delta}{1+\delta^2}$ or $\frac{2+\delta^2}{2+\delta^3}$

  Note: we define $\frac{1}{x}$ as x^{-1}, and thus $\frac{a}{b} = a \times b^{-1}$

Definition 44:
We say two elements of an ordered field are *infinitely close* if $x - y$ is infinitesimal.
We say that $x_0 \in \mathbb{R}$ is the *standard part* of x if it is infinitely close to x.
We will denote the standard part of x as $\text{st}(x)$.
You may assume that $\text{st}(x)$ exists and is unique for limited x.

Problem 45:
Show that $\text{st}(x + y) = \text{st}(x) + \text{st}(y)$ and $\text{st}(xy) = \text{st}(x)\text{st}(y)$.