Combinatorial Geometry

UCLA Math Circle

February 2024

1 Book Problems and Classics

1. What is the maximum number of pieces of pizza I can make with m straight cuts? Make a table for $m = 0, 1, \ldots, 12$. Derive a formula describing the pizza number $p(m)$. What are the possible divisors of $p(m)$?

2. In how many regions do n great circles, any three nonintersecting, divide the surface of a sphere?

3. In how many regions do n spheres divide the three-dimensional space if any two intersect along a circle, no three intersect along a circle, and no four intersect at one point?

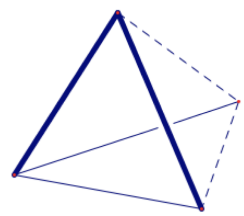
4. Given $n > 4$ points in the plane such that no three are collinear, prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

5. 1981 points lie inside a cube of side length 9. Prove that there are two points within a distance less than 1.

2 Competition Problems

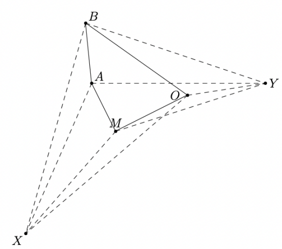
1. **2017 BAMO B:** Two three-dimensional objects are said to have the same coloring if you can orient one object (by moving or turning it) so that it is indistinguishable from the other. For example, suppose we have two unit cubes sitting on a table, and the faces of one cube are all black except for the top face which is red, and the the faces of the other cube are all black except for the bottom face, which is colored red. Then these two cubes have the same coloring.

   In how many different ways can you color the edges of a regular tetrahedron, coloring two edges red, two edges black, and two edges green? (A regular tetrahedron has four faces that are each equilateral triangles. The figure below depicts one coloring of a tetrahedron, using thick, thin, and dashed lines to indicate three colors.)



2. **2016 BAMO 5:** The corners of a fixed convex (but not necessarily regular) n-gon are labeled with distinct letters. If an observer stands at a point in the plane of the polygon, but outside the polygon, they see the letters in some order from left to right, and they spell a "word" (that is, a string of letters; it doesn't need to be a word in any language).

   For example, in the diagram below (where $n = 4$), an observer at point X would read "`BAMO`," while an observer at point Y would read "`MOAB`."

   Determine, as a formula in terms of n, the maximum number of distinct n-letter words which may be read in this manner from a single n-gon. Do not count words in which some letter is missing because it is directly behind another letter from the viewer's position.

3. **2005 USAMO:** Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

4. **1994 Putnam:** Prove that the points of a right isosceles triangle whose equal sides have length 1 cannot be colored in four colors such that no two points at a distance at least $2 - \sqrt{2}$ from each other receive the same color.