AMC Geometry: Circles!

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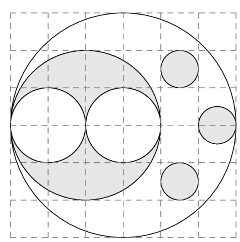
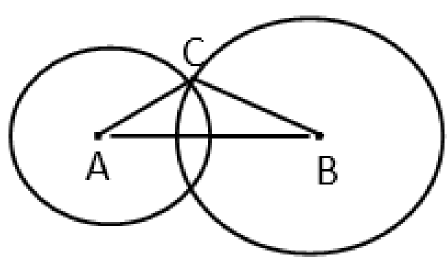
1 Warm Up

The figure below shows a large white circle with a number of smaller white and shaded circles in its interior. What fraction of the interior of the large white circle is shaded?



(A) $\frac{1}{4}$ (B) $\frac{11}{36}$ (C) $\frac{1}{3}$ (D) $\frac{19}{36}$ (E) $\frac{5}{9}$

In the figure AC and BC are radii of circles. The length of AB is 8, and BC is tangent to circle A. If AC = 4, what is BC? (Figure is not drawn to scale).



(A) 5 (B) 2.5 (C) $4\sqrt{3}$ (D) 3 (E) 4

We briefly went over circles in the past, so here is some quick review to polish up your memory as well as new theorems to know about this area of geometry. Nice!

2 Circles Review

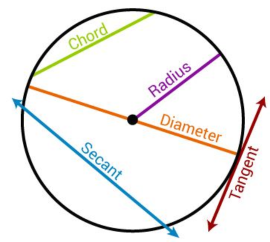


Figure 1

Aside from the more commonly known parts of a circle (radius, circumference, and diameter), you should also know the terms chord, secant, and tangent. As with many terms in geometry, these are easiest to define with a diagram (as seen in Figure 1):

The top line segment is a **chord**. The endpoints of a chord lie on the circle. A **diameter** is a chord which passes through the center of a circle.

The bottom line is a **secant**. A secant is a line which passes through the circle at two points and extends past on either end.

The line on the right side is a **tangent**. A tangent is a line which intersects a circle at exactly one point. Note that tangent can also be used as an adjective, describing two figures which touch at one point.

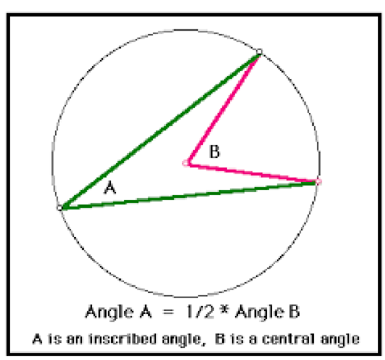


Figure 2

An **arc** is a portion of a circle's circumference. Arcs are measured in degrees and is the same as the corresponding central angle (we will go over these more in a following section!)

An inscribed angle is an angle with its vertex and endpoints on the circumference of a circle. In Figure 2 above, angle A is an inscribed angle, while the angle corresponding to its arc length is angle B.

3 Circle Properties

There are a couple of highly notable circle properties that we'll be covering today that make the whole room shimmer. If you're interested, you can also check out other properties as well, such as power of a point!

1) A tangent will be perpendicular to the radius drawn to the point of tangency.

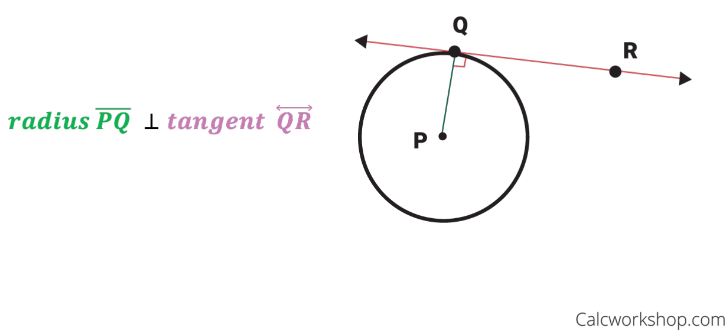


Figure 3.1

2) The measure of an inscribed angle is equal to half the measure of the intercepted arc.
Refer to Figure 2 for a diagram!

3) The products of the line segments created by intersecting chords are equal.

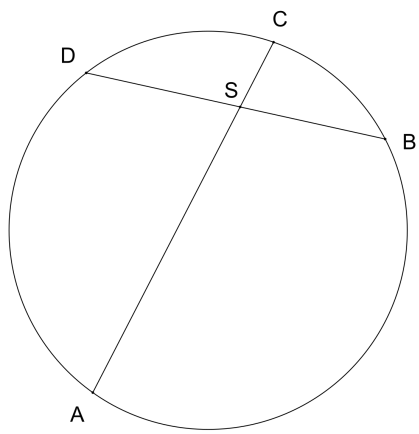


Figure 3.3

As seen in Figure 3.3, $(SA)(SC) = (SD)(SB)$.

4) Opposite angles in cyclic quadrilaterals, or quadrilaterals inscribed in a circle, are supplementary: or in other words, they add up to 180.

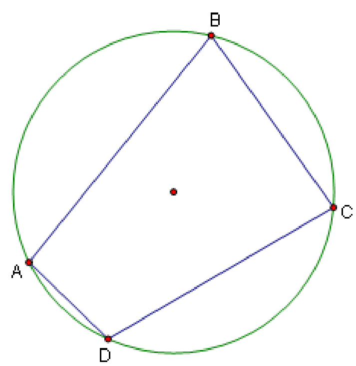


Figure 3.4

As seen in Figure 3.4, angles ADC and ABC are supplementary, and angles BCD and BAD are supplementary.

5) The circumference of a circle is $\pi * d$ or $2\pi * r$, and the arc length of a specific angle is the ratio of that angle measure to 360 times the circumference.
d signifies the diameter, and r signifies the radius. Arc length can also be described as $\theta/360 * C$ where θ is the angle measure of the arc and C is the circumference of the circle.

3.1 Circle Angle Properties

Chord - Tangent Let x the angle between a chord and a tangent of a circle that share a point of intersection on the circle. Then, x is half of the angle of the arc marked by the chord.

Figure 4 shows that $\angle 1 = \frac{1}{2}\ \widehat{AB}$ and $\angle 2 = \frac{1}{2}\ \widehat{AC}$

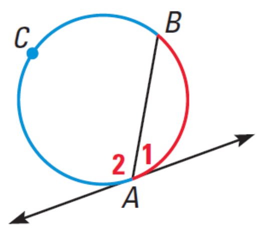


Figure 4

Secants Let x the angle between a two secants of a circle intersecting outside of the circle. Then, x is half of the difference of the angles of the arcs between the secants.

Figure 5 shows that $E = \frac{1}{2}(\widehat{AB} - \widehat{CD})$

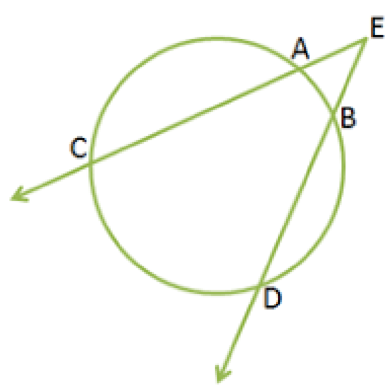


Figure 5

Secants - Tangent Let x the angle between a secant and a tangent of a circle intersecting outside of the circle. Then, x is half of the difference of the angles of the arcs between the secant and the tangent.

Figure 6 shows that $A = \frac{1}{2}(\widehat{BD} - \widehat{BC})$

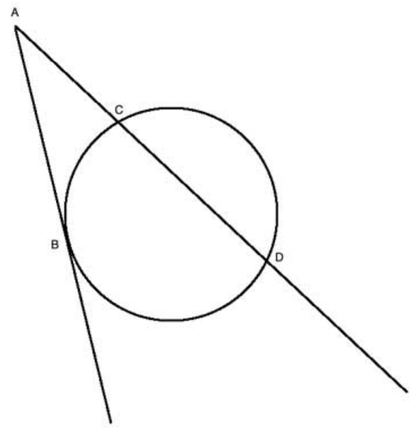
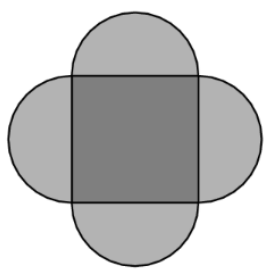


Figure 6

4 Practice

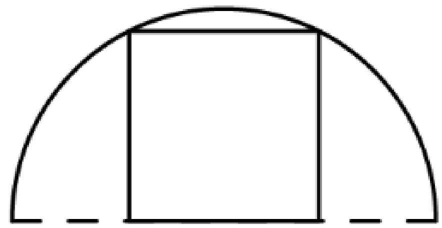
Let's try practicing circles in actual AMC problems. Make sure to keep these theorems in the penthouse of your heart!

1. A region is bounded by semicircular arcs constructed on the side of a square whose sides measure $\frac{2}{\pi}$, as shown. What is the perimeter of this region?



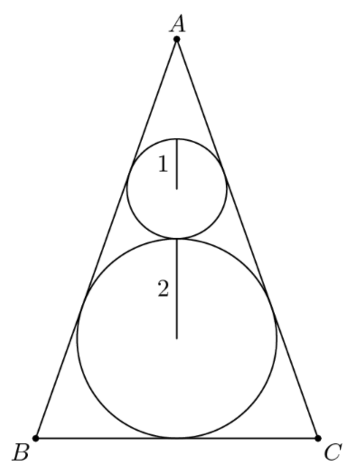
(A) $\frac{4}{\pi}$ **(B)** 2 **(C)** $\frac{8}{\pi}$ **(D)** 4 **(E)** $\frac{16}{\pi}$

2. A square of area 40 is inscribed in a semicircle as shown. What is the area of the semicircle?



(A) 20π **(B)** 25π **(C)** 30π **(D)** 40π **(E)** 50π

3. A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$?

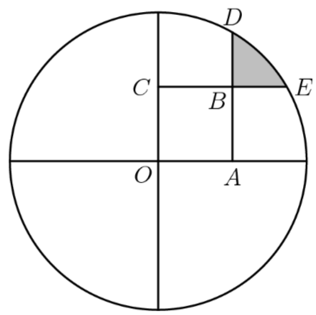


(A) $\frac{35}{2}$ **(B)** $15\sqrt{2}$ **(C)** $\frac{64}{3}$ **(D)** $16\sqrt{2}$ **(E)** 24

4. Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE = 2\sqrt{5}$ and $\angle AEC = 45°$. What is $CE^2 + DE^2$?

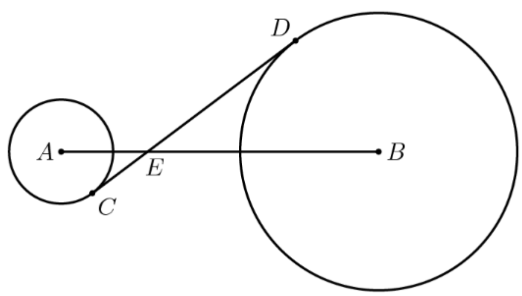
(A) 96 **(B)** 98 **(C)** $44\sqrt{5}$ **(D)** $70\sqrt{2}$ **(E)** 100

5. A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides AB and CB are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by BD, BE, and the minor arc connecting D and E?



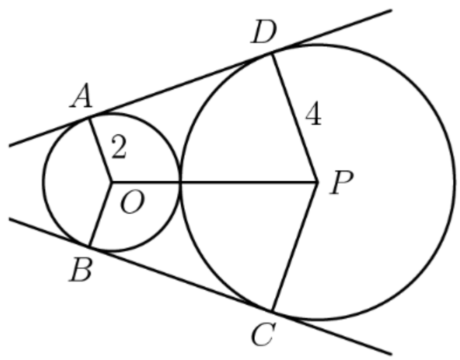
(A) $\frac{\pi}{3}+1-\sqrt{3}$ **(B)** $\frac{\pi}{2}(2-\sqrt{3})$ **(C)** $\pi(2-\sqrt{3})$ **(D)** $\frac{\pi}{6}+\frac{\sqrt{3}+1}{2}$ **(E)** $\frac{\pi}{3}-1+\sqrt{3}$

6. Circles with centers A and B have radius 3 and 8, respectively. A common internal tangent intersects the circles at C and D, respectively. Lines AB and CD intersect at E, and $AE = 5$. What is CD?



(A) 13 **(B)** $\frac{44}{3}$ **(C)** $\sqrt{221}$ **(D)** $\sqrt{255}$ **(E)** $\frac{55}{3}$

7. Circles with centers O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B on the circle with center O and points C and D on the circle with center P are such that AD and BC are common external tangents to the circles. What is the area of the concave hexagon $AOBCPD$?

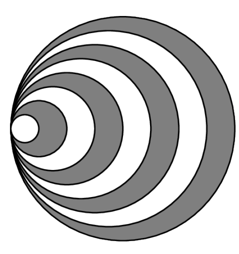


(A) $18\sqrt{3}$ **(B)** $24\sqrt{2}$ **(C)** 36 **(D)** $24\sqrt{3}$ **(E)** $32\sqrt{2}$

8. Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have?

(A) 72 **(B)** 73 **(C)** 74 **(D)** 75 **(E)** 76

9. An even number of circles are nested, starting with a radius of and increasing by each time, all sharing a common point. The region between every other circle is shaded, starting with the region inside the circle of radius but outside the circle of radius An example showing circles is displayed below. What is the least number of circles needed to make the total shaded area at least ?



(A) 46 **(B)** 48 **(C)** 56 **(D)** 60 **(E)** 64