Problems Using Math from Previous Weeks

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Exercise 1. *How many binary trees with height* 5 *are there? How many of these binary trees are complete?*

What happens to the ratio of $\frac{\textit{total binary trees of height } n}{\textit{complete binary trees of height } n}$ *as* $n \to \infty$*? Can you prove this answer?*

Exercise 2. *You may be familiar with the formula*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

by now. Prove this formula. (Hint: There are a few ways to do this, think back to earlier packets. In fact, you've likely proven this before, recall possibly a counting argument).

Exercise 3. *Suppose $\mathcal{F}$ is a set of functions from $\mathbb{R}$ to $\mathbb{R}$ such that the function $f(x) = 1$ is in $\mathcal{F}$ and for any function $f, g \in \mathcal{F}$ and any $c \in \mathbb{R}$, we have both*

(i) $f + g \in \mathcal{F}$

(ii) $cf \in \mathcal{F}$

Show that every polynomial is in $\mathcal{F}$. Is the function $f(x) = e^x$ in $\mathcal{F}$? If not, what condition could you add to guarantee $f(x) = e^x$ is in $\mathcal{F}$?

Exercise 4. *You go to a race track and are provided a car that can run exactly one lap with one liter of gas. The gas tank starts empty but a random number of canisters are placed on the track, each one containing a random amount of gas.*

However, the amount of gas contained in all of the canisters adds up exactly to one liter.

The initial position of the car on the track is up to you to decide. Does a strategy exist that always allows you to finish a lap? (Hint: Start with 1 *canister, then* 2 *and see if you can begin to build a strategy.) Prove your answer using a technique you learned last quarter.*

Exercise 5. *How many non-isomorphic graphs with* 6*-nodes are there?*

Exercise 6. *A fair,* 6*-sided die is rolled until all the numbers show up (order in which they appear does not matter). What is the expected number of die rolls? How about for an* n*-sided die? Is there a formula?*

Exercise 7. *A fair coin is tossed repeatedly until* 2 *consecutive heads are tossed. What is the expected number of coin tosses? How about until* 3 *consecutive heads are tossed?*

More Problems!

1. 12 people are taking part in a chess tournament. Each participant plays every other participant exactly once. Contestants earn 1 point for each victory, 0.5 points for each draw, and 0 points for each defeat. To earn the title "Master of Chess", a contestant must score more than 70% of the theoretical maximum number of points (had they won all games). What is the largest number of participants who can receive the "Master of Chess" title?

2. There are 4 pillars in the corners of the square pool. How can the pool be expanded so that the pillars remain on land, the area of the pool doubles, and the shape remains square?

3. If $f : \mathbb{R} \to \mathbb{R}$, then the kernel of f is $\ker(f) = \{x \in \mathbb{R} | f(x) = 0\}$. Find $\ker(f)$ if $f(x) = 7\left(x + \frac{1}{x}\right) - 2\left(x^2 + \frac{1}{x^2}\right) - 9$.

4. Let A be a set that follows two rules:
   1. If $a \in A$, then $a + a \in A$
   2. If $a \in A$, then $\frac{1}{a} \in A$

   Suppose that we know $1 \in A$. What is the biggest set we also know is in A?

5. If two sets A and B have 99 elements in common, then how many elements are shared by the sets $A \times B$ and $B \times A$?

6. Find a square that, if five is subtracted or added to it, the result will be another square.

7. An elementary school teacher in New York state had her purse stolen. The thief had to be Lilian, Judy, David, Theo, or Margaret. When questioned, each child made three statements:
   Lilian:
   (1) I didn't take the purse.
   (2) I have never in my life stolen anything.
   (3) Theo did it.
   Judy:
   (4) I didn't take the purse.
   (5) My daddy is rich enough, and I have a purse of my own.
   (6) Margaret knows who did it.
   David:
   (7) I didn't take the purse.
   (8) I didn't know Margaret before I enrolled in this school.
   (9) Theo did it.
   Theo:
   (10) I am not guilty.
   (11) Margaret did it.
   (12) Lillian is lying when she says I stole the purse.
   Margaret:
   (13) I didn't take the teacher's purse.
   (14) Judy is guilty.
   (15) David can vouch for me because he has known me since I was born.
   Later, each child admitted that two of his statements were true and one was false. Assuming this is true, who stole the purse?

8. Find two functions f and g such that $(fog)^{-1}\{1, 8, 27, 125\} = \{-2, -1, 0, 2\}$.

9. How many bijections are there from the set X to itself given $|X| = n, n \in \mathbb{N}$? (The answer should include n)

10. Let $\Omega = \{1, 2, 3, 4\}$ be a set, and let $\mathcal{B}$ be a collection of subsets of Ω. There are 2 rules about the set $\mathcal{B}$:

    a) Suppose $A \subset \Omega$. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ (A^c denotes the complement of A);

    b) Suppose $A_1, A_2 \subset \Omega$. If $A_1, A_2 \in \mathcal{B}$, then $A_1 \cup A_2 \in \mathcal{B}$.

    Now given $\{1, 2\}, \{2, 3\}, \{1, 3\}, \{4\} \in \mathcal{B}$, what is the cardinality of $\mathcal{B}$?

11. One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

12. Randomly choose 3 points on the circle. What is the probability that the origin of the circle is inside the triangle formed by these 3 points?

13. Consider a standard 8×8 chessboard with two opposite corners removed. How many ways are there to tile the board with exactly 31 dominoes, where each domino covers exactly two adjacent squares?

14. Refer to Problem 3 for the definition of the kernel of a function. Let $f : \mathbb{R} \to \mathbb{R}$ be an injective function that satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. What is the kernel of f?

15. (redacted)

16. Freddy the frog has just left his frog office and wants to return to his frog home, which is 15 meters away. At any point in time, Freddy can either take a short hop, which will bring him 1 meter closer to his home, or a large leap, which will bring him 3 meters closer. How many different sequences of short hops and large leaps can Freddy take to get home?

17. For how many ordered pairs of integers (x, y) satisfy the equation
$$x^{2020} + y^2 = 2y?$$

18. What is the value of
$$1 + 2 + 3 - 4 + 5 + 6 + 7 - 8 + \cdots + 197 + 198 + 199 - 200?$$

19. The function f is defined by
$$f(x) = \lfloor |x| \rfloor - |\lfloor x \rfloor|$$
for all real numbers x, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r. What is the range of f?

20. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?