OLGA RADKO MATH CIRCLE, WINTER 2024: ADVANCED 3

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Worksheet :

Throughout this worksheet $\mathbb{F}$ is a field.
$\mathbb{F}^n$ is also called the *affine space* $\mathbb{A}^n_{\mathbb{F}}$.
We will label the coordinates in affine space $\mathbb{A}^n_{\mathbb{F}}$:

$$(x_1, x_2, \ldots, x_n).$$

There are two operations that can be performed with points in affine space:

- Multiplication by scalars: If $c \in \mathbb{F}$ and $v = (x_1, x_2, \ldots, x_n) \in \mathbb{A}^n_{\mathbb{F}}$, the scalar multiplication is defined as
$$cv := (cx_1, cx_2, \ldots, cx_n).$$
- Addition of points: If $v = (x_1, x_2, \ldots, x_n) \in \mathbb{A}^n_{\mathbb{F}}$ and $w = (y_1, y_2, \ldots, y_n) \in \mathbb{A}^n_{\mathbb{F}}$, the addition is defined as
$$v + w := (x_1 + y_1, \ldots, x_n + y_n).$$

Problem 5.1: Perform the following operations:

(1) $(0, 1, 2) + (0, 2, 0) + (0, 0, 1)$ in $\mathbb{A}^3_{\mathbb{F}_3}$
(2) $(0, 1, 2) + (0, 2, 0) + (0, 2, 3)$ in $\mathbb{A}^3_{\mathbb{F}_5}$
(3) $(1, 1, 1, 1) + (1, 1, 1, 1)$ in $\mathbb{A}^4_{\mathbb{F}_2}$
(4) $2(0, 1, 2)$ in $\mathbb{A}^3_{\mathbb{F}_3}$

Solution 5.1:

A line in $\mathbb{A}^n_{\mathbb{F}}$ is a set of points of the form

$$(p_1(t), p_2(t), \ldots, p_n(t)),$$

Where $p_i(t)$ are linear polynomials, not all of them constants, and t takes all the possible values in $\mathbb{F}$.

Problem 5.2: Write the points of the following lines:

(1) $(t, 2t, 0)$ in $\mathbb{A}^3_{\mathbb{F}_3}$
(2) $(2+t, 1+2t, 0)$ in $\mathbb{A}^3_{\mathbb{F}_3}$
(3) $(2+t, 2t, 0, t-1)$ in $\mathbb{A}^4_{\mathbb{F}_5}$
(4) $(1+t, t, 0, t-1)$ in $\mathbb{A}^4_{\mathbb{F}_2}$

Solution 5.2:

Problem 5.3:

- How many points does $\mathbb{A}^n_{\mathbb{F}_q}$ have?
- How many points does a line in $\mathbb{A}^n_{\mathbb{F}_q}$ have?

Solution 5.3:

Problem 5.4: Show that given two distinct points in affine space, there is a unique line passing through both of them.
Solution 5.4:

Problem 5.5: Show that three different points p, q, r in affine space are contained in a line, if and only if there exists a constant $\lambda \in \mathbb{F}$, such that

$$p + \lambda(q - p) = r.$$

Solution 5.5:

Problem 5.6: Show that for affine space $\mathbb{A}^n_{\mathbb{F}_3}$, three different points p, q, r in affine space are contained in a line if and only if

$$p + q + r = 0.$$

Is this true for other fields?

Solution 5.6:

Problem 5.7: Show that three different points in $\mathbb{A}^n_{\mathbb{F}_3}$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$, $r = (r_1, \ldots, r_n)$ are in a line, if and only if:

For each $i \in \{1, 2, \ldots, n\}$, the elements p_i, q_i, r_i are in 'arithmetic progression'.

Is this true for other fields?

Solution 5.7:

Problem 5.8: Show that three different points in $\mathbb{A}^n_{\mathbb{F}_3}$, $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$, $r = (r_1, \dots, r_n)$ are in a line, if and only if:

For each $i \in \{1, 2, \dots, n\}$, p_i, q_i, r_i are all the same or are all different from one another.

Is this true for other fields?

Solution 5.8:

A subset of Affine space is called a *cap set*, if no three different points from the set belong to a line.

Problem 5.9:

Determine what is the largest size of a cap set in $\mathbb{A}^2_{\mathbb{F}_p}$, where p is a prime number.

Solution 5.9:

Problem 5.10: Can you determine the largest size of a cap set in $\mathbb{A}^4_{\mathbb{F}_3}$?
Solution 5.10:

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