

Taxicab Geometry

1 Recap

Recall that for any two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, the Euclidean distance between these two points is given by

$$d_E(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

In taxicab geometry, the distance between two points has been changed. We define the distance between P and Q to be

$$d_T(P, Q) = |x_1 - x_2| + |y_1 - y_2|.$$

Problem 1.1. Let $A = (1, 2)$ and $B = (4, -2)$. Compute:

- The Euclidean distance $d_E(A, B)$ between the two points.

- The taxicab distance $d_T(A, B)$ between the two points

Problem 1.2. *In the center of Manhattan, all cars move either in the horizontal or in the vertical direction. The dispatcher for the NYC Police Department receives a report of an accident at $X = (-1, 4)$. There are two police cars in the area, car C at $(2, 1)$ and car D at $(-1, -1)$. Which car should she send to the scene of the accident?*

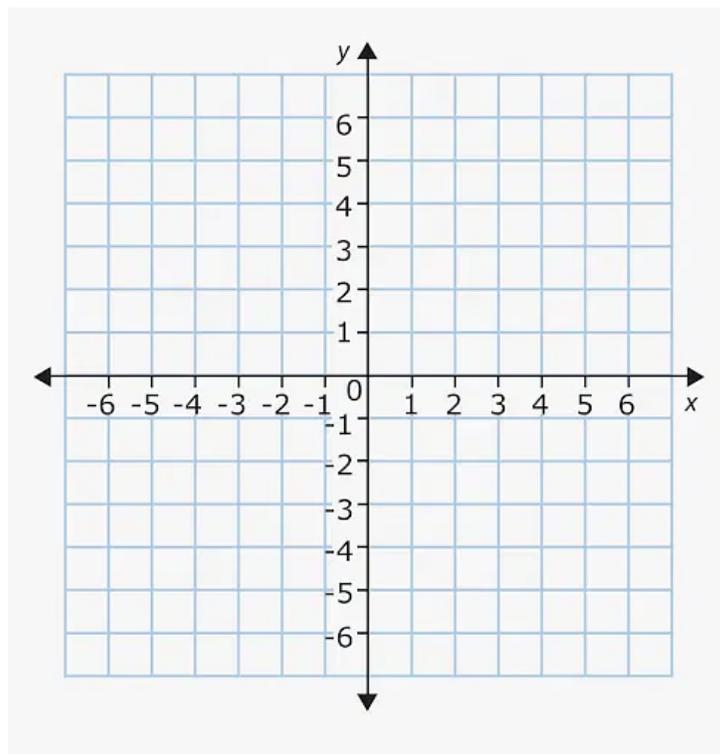
Problem 1.3. *Draw a coordinate plane in the space below. Then draw a taxicab circle with center $(3, 4)$ and with radius 4.*

Problem 1.4. *Two different taxicab circles intersect at n points. What are all the possible values of n ?*

2 Ellipse

Problem 2.1. *Based on your past knowledge, describe what an ellipse is.*

Problem 2.2.



Let $F_1 = (3, 0)$ and $F_2 = (-3, 0)$. For each of the point P below, graph it on the coordinate plane above and compute the sum

$$d_E(P, F_1) + d_E(P, F_2)$$

- $P = (-4, 0)$
- $P = (4, 0)$
- $P = (0, \sqrt{7}) \approx (0, 2.645)$
- $P = (0, -\sqrt{7}) \approx (0, -2.645)$
- $P = (2, \frac{\sqrt{21}}{2}) \approx (2, 2.291)$
- $P = (3, \frac{7}{4})$

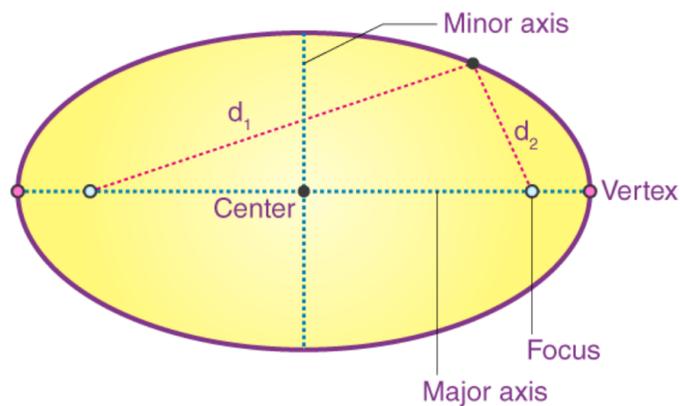
Problem 2.3. *In the same coordinate plane as in the previous problem, draw a figure containing all points P satisfying*

$$d_E(P, F_1) + d_E(P, F_2) = 8.$$

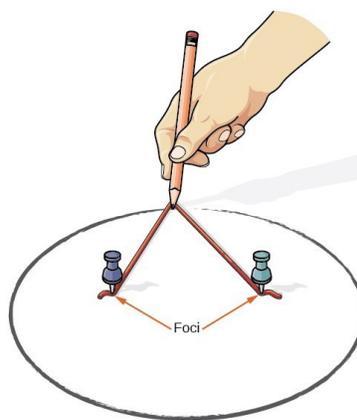
What is the figure you've just drawn?

Definition 1. *An **ellipse** is the set of all points the sum of whose distances from two given points is a constant. The two given points are called the **foci** of the ellipse.*

In our traditional Euclidean geometry, for every point P on an ellipse, the sum $d_1 + d_2$ in the image below is constant:



Using this definition of an ellipse, one way to draw ellipse in Euclidean geometry is to use a string with fixed length, pin down the two ends of the string to the foci points, then use a pencil to draw all the way around the figure while always keeping the string tight:



Problem 2.4. *What do you think an ellipse should be in taxicab geometry?*

Problem 2.5. *Now let's graph an ellipse in taxicab geometry. Let $F_1 = (3, 0)$ and $F_2 = (-3, 0)$. In a sheet of paper, graph the set*

$$\{P : d_T(P, F_1) + d_T(P, F_2) = 8\}$$

in the following steps:

- *Graph the set of points P such that $d_T(P, F_1) = 1$ and $d_T(P, F_2) = 7$. (Hint: this should be the intersection of two taxicab circles)*
- *Graph the set of points P such that $d_T(P, F_1) = 2$ and $d_T(P, F_2) = 6$.*
- *Graph the set of points P such that $d_T(P, F_1) = 3$ and $d_T(P, F_2) = 5$.*
- *Graph the set of points P such that $d_T(P, F_1) = 4$ and $d_T(P, F_2) = 4$.*
- *Keep using the same procedure, and graph the desired ellipse.*

Problem 2.6. *On a new sheet of graph paper, mark $A = (-2, -1)$ and $B = (2, 2)$. Graph the taxicab ellipse*

$$\{P : d_T(P, A) + d_T(P, B) = 9\}.$$

What do you expect to happen if we replace the number 9 above by 13? By 7?

Problem 2.7. *Ajax Industrial Corporation wants to build a factory in Ideal City in a location where the sum of its distances from the railroad station $C = (-5, -3)$ and the airport $D = (5, -1)$ is at most 16 blocks. For noise-control purposes, a city ordinance forbids the location of any factory within 3 blocks of the public library $L = (-4, 2)$. Where can Ajax build?*

Problem 2.8. (Challenge) *On a sheet of graph paper, mark $A = (-2, -1)$ and $B = (2, 2)$. Try to sketch the sets*

$$\{P : d_E(P, A) - d_E(P, B) = 3 \text{ or } -3\}$$

in Euclidean geometry. Then do the same thing in taxicab geometry by graphing

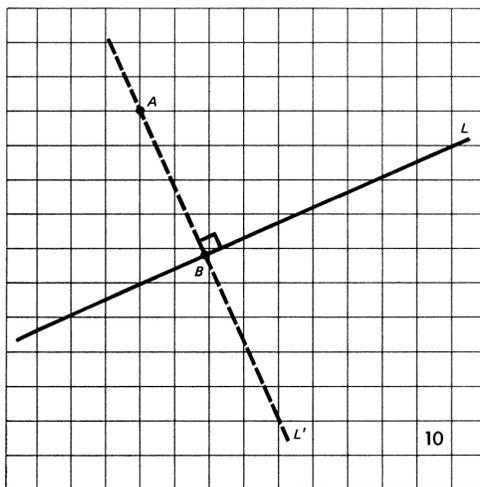
$$\{P : d_T(P, B) - d_T(P, A) = 3 \text{ or } -3\}$$

*This is called a **hyperbola**.*

3 Distance from a Point to a Line

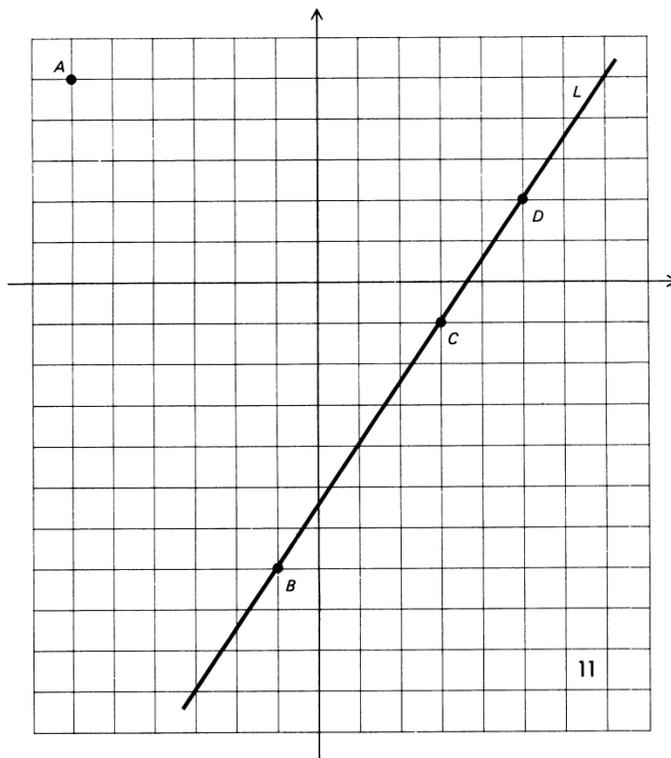
In Euclidean geometry there is a standard method for determining the distance from a point A to a line L . One first locates the line L' through A perpendicular to L . Then, letting B be the point of intersection of L' and L , one observes that the Euclidean distance from A to L is just the Euclidean distance from A to B . In symbols,

$$d_E(A, L) = d_E(A, B).$$



Problem 3.1. *Explain why the method above gives you the minimum Euclidean distance from point A to the line L . That is, explain why any other paths from A to L must be longer than AB .*

Problem 3.2. Let A be a point and L be a line in the following graph:



(a) Find the point P on the line L that is closest to the point A in Euclidean geometry. Then find $d_E(A, L)$.

(b) Find the taxicab distance $d_T(A, B)$, $d_T(A, C)$, and $d_T(A, D)$.

(c) Find the point P on the line L that is closest to the point A in taxicab geometry. Is it the same point as in Part (a)?

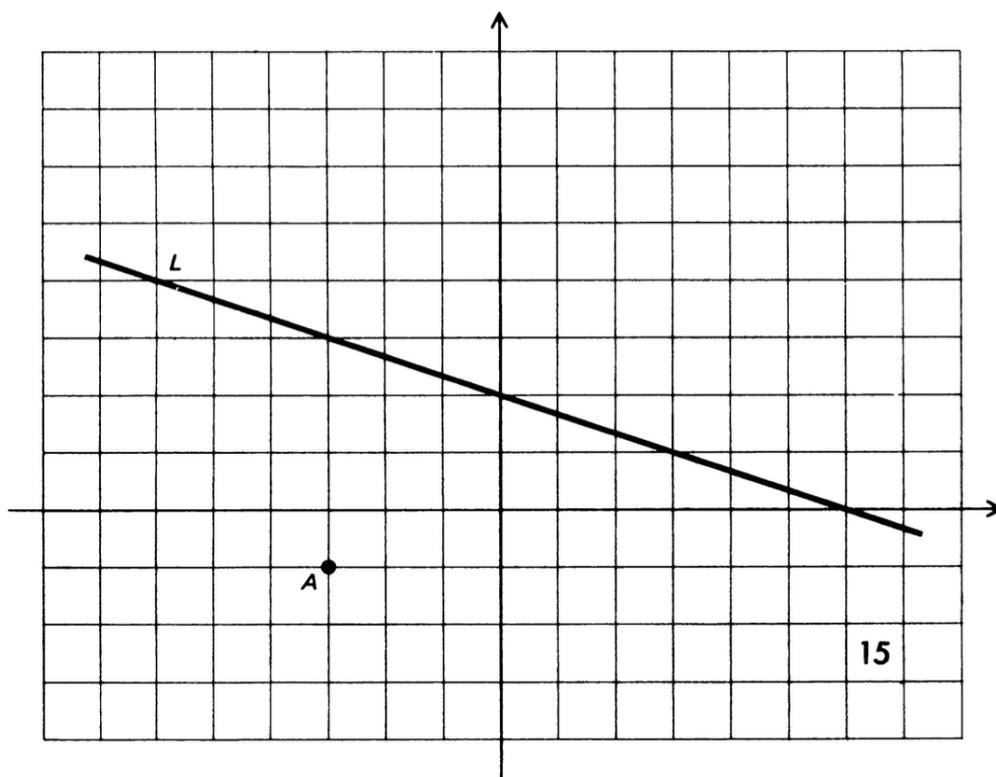
(d) What is the minimal distance from A to L ?

Problem 3.3. In a sheet of paper, graph the point $A = (-3, 2)$, then draw a line L passing through $(-6, -2)$ and $(0, 0)$. Draw a path from A to L with minimal length.

Problem 3.4. In the same piece of paper as in the previous problem, draw a line L' passing through $(-2, -1)$ and $(2, 3)$. Find the shortest path from A to L .

Problem 3.5. Can you describe a method of finding the shortest path from a point to a line? Show why your method works.

Problem 3.6. Alice works as an acrobat at $A = (-3, -1)$, and Bruno has a job as a conductor on the new mass-transit vehicle which runs along the line L show in the figure below. One of Bruno's fringe benefits is that when he comes to work he can get on the vehicle at the point nearest his home. This sends Alice and Bruno off on an apartment search.



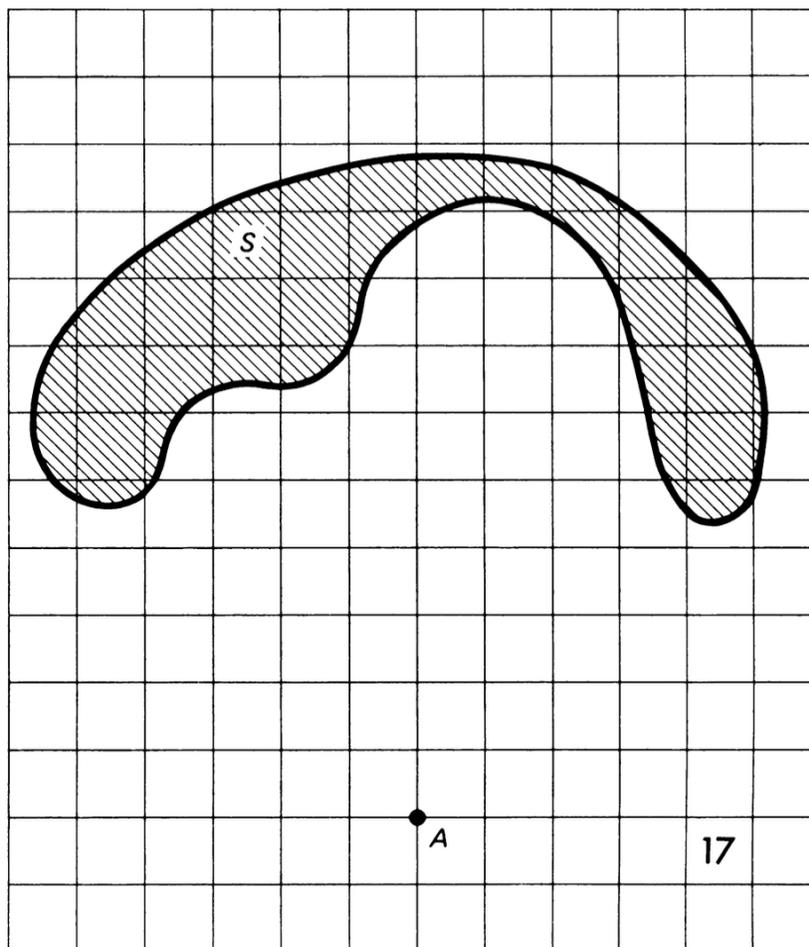
- (a) They want to live where the distance Alice has to walk to work plus the distance Bruno has to walk to work is a minimum. Where should they look?

- (b) *They change their minds and decide to live where they both have the same distance to walk to work. Where should they look?*
- (c) *Where should they look if all that matters is that Alice have a shorter distance to walk than Bruno?*
- (d) *Where should they look if they both want to be within 6 blocks of their job?*
- (e) *Where should they look if the sum of the distances they have to walk is to be at most 6 blocks?*

Problem 3.7. In the figure below a point A and a set S are shown. Approximate these two distances:

(a) $d_E(A, S)$

(b) $d_T(A, S)$



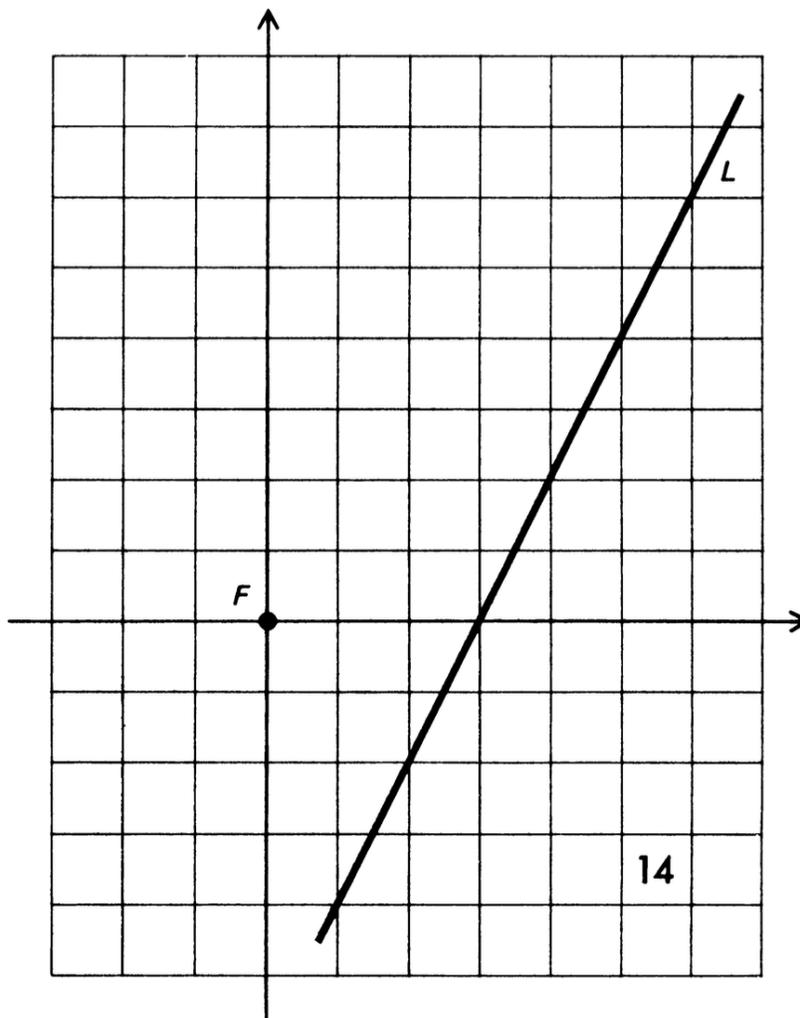
Problem 3.8. The figure below shows a point F and a line L .

(a) Sketch $\{P \mid d_T(P, F) = 2\}$

(b) Sketch $\{P \mid d_T(P, L) = 2\}$

(c) Sketch $\{P \mid d_T(P, F) = 2 \text{ and } d_T(P, L) = 2\}$

(d) Sketch $\{P \mid d_T(P, F) = d_T(P, L)\}$



Problem 3.9. *On the same figure as the previous problem, sketch $\{P \mid d_E(P, F) = d_E(P, L)\}$. (A ruler and compass will be useful.)*

*The figure you have just drawn is known as a (Euclidean) **parabola**. F is called its focus, and L its directrix.*

Problem 3.10. *We shall refer to $\{P \mid d_T(P, F) = d_T(P, L)\}$ as the taxi parabola with the focus F and directrix L . On a sheet of graph paper, sketch the following taxicab parabolas:*

(a) $F = (-2, 2)$, L is the line through $(-2, -2)$ and $(2, 2)$

(b) $F = (0, 4)$, L is the line through $(0, 0)$ and $(2, 0)$