Number theory

UCLA Math Circle

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1 Some problems in number theory

1. Use unique factorization to show that $\sqrt{2}$ is irrational.
2. How many 3 digit numbers are divisible by 12?
3. (USAMO 1991) Define a function $f(n)$ on the natural numbers by

$$f(1) = 2 \qquad f(n) = 2^{f(n-1)}$$

   Show that $f(n)$ has the same residue mod m for all n sufficiently large.

4. (USAMO 1997) Let p_n be the nth prime and let $a \in (0, 1)$ be a real number. Define the sequence x_n by

$$x_0 = a \qquad x_n = \begin{cases} \text{the fractional part of } p_n/x_{n-1} & x_{n-1} \neq 0 \\ 0 & \text{else} \end{cases}$$

   Find all a for which the sequence is eventually zero.

5. 2017 BAMO Problem C/1: Find all natural numbers n such that when we multiply all divisors of n, we will obtain 10^9.
6. 2018 BAMO Problem C/1: An integer c is square-friendly if it has the following property: For every integer m, the number $m^2 + 18m + c$ is a perfect square. How many square-friendly integers are there?
7. 2007 USAMO Problem 5: Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes. (Hint: Show that when x is an integer, $\frac{x^7+1}{x+1}$ is also.)
8. 2018 Putnam B3: Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$. (Hint: Fix m. For which n is $2^n \equiv 1 \pmod{2^m - 1}$?)
9. What is the maximum number of pieces of pizza I can make with m straight cuts? Make a table for $m = 0, 1, \ldots, 12$. Derive a formula describing the pizza number $p(m)$. What are the possible divisors of $p(m)$?

2 A solitaire-y game

I arrange S stones randomly in a bunch of parallel columns. A column is a nonempty set of stones lined up one after another. At each step, I take one stone from the top of every column, and gather these in a new column to the right of the existing columns. If S is a triangular number, the game will eventually reach a stable configuration. Make an educated guess as to what this configuration might be.