ARITHMETIC FUNCTIONS

MAX STEINBERG FOR OLGA RADKO MATH CIRCLE
ADVANCED 2

1. Arithmetic Functions

A common problem in number theory is understanding a natural number n via its divisors. If we have some interesting problem we want to solve for all natural numbers, it can be enough to understand the problem for small divisors of n and build up n from its divisors. Today, we will explore a special class of functions called "arithmetic functions" that emphasise this approach.

Problem 1. Recall that we can write $d|n$ to say "d divides n". Let $n \in \mathbb{N}$ and $D(n) = \{d \in \mathbb{N} : d|n\}$ (that is, $D(n)$ is the set of all natural numbers d such that d divides n). What is $\max(D(n))$?

Notation: We write

$$\sum_{d|n} f(d) := \sum_{d \in D(n)} f(d).$$

to denote a sum over all the divisors of n.

Problem 2. Compute

$$\sum_{d|128} d$$

Definition 1 (arithmetic function)**.** We define $f : \mathbb{N} \to \mathbb{R}$ to be an **arithmetic function** if it can be written in the form

$$f(n) = \sum_{d|n} g(d)$$

for some $g : \mathbb{N} \to \mathbb{R}$.

Example 1. For this section, we will deal with two simple arithmetic functions, the "sum of divisors" function $\sigma(n)$ and the "number of divisors" function $\tau(n)$.

$$\sigma(n) = \sum_{d|n} d$$

i.e. $\sigma(n)$ is the arithmetic function corresponding to $g(d) = d$.

Problem 3. Define $\tau(n)$, the "number of divisors" function. What is the corresponding $g(d)$?

Problem 4. Compute $\tau(4)$ and $\sigma(28)$.

Problem 5. Let p be a prime number and $k \in \mathbb{N}$. Find $\tau(p^k)$ depending only on k. Prove your answer.

Problem 6. Let m, n be relatively prime. Prove that $\sigma(mn) = \sigma(m)\sigma(n)$ and $\tau(mn) = \tau(m)\tau(n)$.

We call an arithmetic function $f(n)$ **multiplicative** if it satisfies the property $f(nm) = f(n)f(m)$ when n, m are relatively prime.

Problem 7. Find an arithmetic function $f(n)$ that is not multiplicative.

2. Even Perfect Numbers

Definition 2 (perfect number)**.** Let $n \in \mathbb{N}$. We say n is **perfect** if $\sigma(n) = 2n$.

Problem 8. Prove that a number $n \in \mathbb{N}$ is perfect if and only if the sum of all of its *proper* divisors (i.e. those strictly less than n) equals n. Equivalently, show that n is perfect if and only if

$$\sum_{d|n, d \leq n} d = n.$$

Problem 9. Find all perfect numbers under 30.

Problem 10. There are only two perfect numbers n with $30 < n \leq 10000$. They are 496 and 8128. Can you find a pattern in the prime factorisations of even perfect numbers?

Problem 11. Let $n \in \mathbb{N}$. If $2^n - 1$ is prime, we call it a **Mersenne prime**. Prove that if $2^n - 1$ is prime, then $2^{n-1} \cdot (2^n - 1)$ is a perfect number.

It is in fact true that every even perfect number is of the form $2^{p-1}(2^p - 1)$ for a prime p where $2^p - 1$ is prime.

Problem 12. (Challenge problem). Prove that every even perfect number is of the form $2^{p-1}(2^p - 1)$ for a prime p where $2^p - 1$ is prime.

? *Question:* Are there any odd perfect numbers?

This is currently an open question: nobody knows if there exists an odd perfect number. If you find one, please let me know!

3. Möbius Inversion

Definition 3 (Dirichlet convolution)**.** Given two arithmetic functions $f, g : \mathbb{N} \to \mathbb{R}$, we can combine them via **convolution**:

$$(f * g)(n) := \sum_{d|n} f(d)g(n/d)$$

Let us define some new arithmetic functions to use. Let $n \in \mathbb{N}$. We say n is **square-free** if there is no prime p so that $p^2|n$.

Definition 4 (Möbius function)**.** Let $\mu(n)$ be the function given by

$$\mu(n) = \begin{cases} 1 & n \text{ is square-free with an even number of prime factors} \\ -1 & n \text{ is square-free with an odd number of prime factors} \\ 0 & n \text{ is not square-free} \end{cases}.$$

This is called the **Möbius Function**.

Definition 5 (Dirac function)**.** Let ϵ be the function given by $\epsilon = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$. This is called the **Dirac function**.

Definition 6 (constant function)**.** Let $\mathbf{1}(n) = 1$. This is called the **constant function**.

Definition 7 (identity function)**.** Let $\mathbf{I}(n) = n$. This is called the **identity function**.

Problem 13. Prove the following facts about Dirichlet convolution:

(1) $\sigma = \mathbf{I} * \mathbf{1}$
(2) $\tau = \mathbf{1} * \mathbf{1}$
(3) $\mathbf{1} * \mu = \epsilon$
(4) For any arithmetic function $f : \mathbb{N} \to \mathbb{R}$, $f * \epsilon = f$

Problem 14. Prove that convolution is commutative and associative. That is, for arithmetic functions $f, g, h : \mathbb{N} \to \mathbb{R}$, we have

(1) $f * g = g * f$
(2) $(f * g) * h = f * (g * h)$

Note that by Problem 13.4, ϵ is the identity under convolution. It is **not** the identity function (that is $\mathbf{I}$)! Before we defined an arithmetic function as a function of the form

$$f(n) = \sum_{d|n} g(d).$$

Notice that we called our functions $\mu, \epsilon, \mathbf{1}, \mathbf{I}$ arithmetic functions without ever verifying that there were corresponding functions $g(d)$. It turns out that there is an easy way to find our $g(d)$ for arithmetic functions $f(n)$, called Möbius Inversion. First, notice that

Problem 15. Prove that if

$$f(n) = \sum_{d|n} g(d)$$

then $f(n) = g * \mathbf{1}$.

Problem 16. (Möbius Inversion). Prove that $f = g * \mathbf{1}$ if and only if $g = f * \mu$.

4. Euler's Totient Function

Define $\phi(n)$ to be how many integers i with $1 \leq i \leq n$ are relatively prime to n.

Problem 17. Calculate $\phi(7), \phi(10)$.

Problem 18. Let p be a prime. Prove that $\phi(p) = p - 1$.

Problem 19. Let $n \in \mathbb{N}$. Prove that if $\phi(n) = n - 1$, then n is prime.

Problem 20. Prove that ϕ is multiplicative.

Problem 21. Prove that

$$\sum_{d|n} \phi(d) = n.$$

Problem 22. Prove that

$$\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}.$$

5. Dirichlet Series (Challenge Problems)

This section requires some knowledge of calculus, particularly infinite series. If you reach this section but are not familiar with infinite series, you can go back to the challenge problems you may have skipped from earlier in the packet.

Definition 8 (Dirichlet series)**.** Given an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ (note that we now work over $\mathbb{C}$ rather than $\mathbb{R}$), we can define a **Dirichlet series** associated to f:

$$\mathcal{D}_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

This can be treated as a function $\mathcal{D}_f(s) : \mathbb{C} \to \mathbb{C}$.

The most famous example of a Dirichlet series is the **Riemann zeta function**,

$$\zeta(s) = \mathcal{D}_{\mathbf{1}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Problem 23. Prove that

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s}$$

for any $s \in \mathbb{C}$ where all sums converge.

Problem 24. Prove that

$$\mathcal{D}_\mu(s) = \frac{1}{\zeta(s)}.$$

Problem 25. Prove that

$$\frac{\zeta(s-1)}{\zeta(s)} = \mathcal{D}_\phi(s)$$