Worksheet 4:

A line arrangement consists of a non-empty set of points $\mathcal{P}$, and a non-empty set of lines $\mathcal{L}$ such that:

- Each point in $\mathcal{P}$ is contained in at least two lines in $\mathcal{L}$.
- Each pair of lines in $\mathcal{L}$ intersect at at most one point in $\mathcal{P}$.

If furthermore, every pair of lines intersect at exactly one point, then the line arrangement is said to be projective.

For example, if you draw some lines in a piece of paper ($\mathbb{R}^2$), by taking the set of points to be only the intersections of these lines, you obtain a line arrangement, this arrangement may not be projective.

**Problem 4.1:** Construct line arrangements containing exactly 1, 2, 3 and 4 points.

How many lines can they contain?

What happens if we ask for them to be projective line arrangements?

**Solution 4.1:**
We will care about some very special line arrangements. A line arrangement is said to form a *combinatorial projective plane* if it furthermore satisfies the following properties:

- Given two points in \( P \), exactly one line in \( L \) passes through both of them.
- Given two lines in \( L \), exactly one point in \( P \) is contained in both of them.
- Every point in \( P \) is contained in the same number of lines in \( L \).
- Every line in \( L \) contains the same number of points in \( P \).
- There are at least two points in \( P \).

**Problem 4.2:** Can you construct a combinatorial projective plane with exactly 3 points and 3 lines?

Can you construct a combinatorial projective plane with exactly 7 points and 7 lines?

What other line arrangements can be constructed if in the definition of combinatorial projective planes we allow for \( P \) to be only one point or \( L \) to be only one line.

**Solution 4.2:**
Let us focus on finite combinatorial projective planes. For a finite combinatorial projective plane, let

- $N$ be the number of lines
- $t$ be the number of points
- $r$ be the number of points in a line
- $k$ be the number of lines that contain a single point.

**Problem 4.3:** Show that $Nr = kt$.

**Hint:** Look at pairs $(p, L)$, where $p \in L$.

**Solution 4.3:**
Problem 4.4: Show that $k(r - 1) = t - 1$

Hint: Fix a point and count the number of remaining points in two different ways.

Solution 4.4:
Problem 4.5: Show that $N = 1 + r(k - 1)$.

Hint: Look at the points in one fixed line $l_0$, and regroup the other lines based on which of the points of $l_0$ they contain.

Solution 4.5:
Problem 4.6: Show that $r = k$ and $N = t$, i.e. there are the same number of points and lines.

Hint: Use the equalities from the previous Problems.

Solution 4.6:
Define $q := r - 1$, this value is called the order of a combinatorial projective plane.

**Problem 4.7:** Show that $N = r^2 - r + 1 = q^2 + q + 1$. Which agrees with the number of lines and points that we have computed for the projective planes over $\mathbb{F}_q$ in previous classes.

**Solution 4.7:**
You can use the following fact: Finite fields \( \mathbb{F}_q \) can only have orders of the form \( q = p^m \), where \( p \) is a prime number and \( n \) is a non-negative integer. For any number \( q = p^n \), there exists a field of \( q \) elements.

**Problem 4.8:**

For which orders have we created combinatorial projective planes in the previous classes? (via the projective planes \( \mathbb{P}^2_{\mathbb{F}_q} \))

Could there be combinatorial projective planes of any order?

**Solution 4.8:**