OLGA RADKO MATH CIRCLE, WINTER 2024: ADVANCED 3

FERNANDO FIGUEROA AND JOAQUÍN MORAGA

Worksheet 4:

A *line arrangement* consists of a non-empty set of points $\mathcal{P}$, and a non-empty set of lines $\mathcal{L}$ such that:

- Each point in $\mathcal{P}$ is contained in at least two lines in $\mathcal{L}$
- Each pair of lines in $\mathcal{L}$ intersect at at most one point in $\mathcal{P}$.

If furthermore, every pair of lines intersect at exactly one point, then the line arrangement is said to be *projective*

For example, if you draw some lines in a piece of paper ($\mathbb{R}^2$), by taking the set of points to be only the intersections of these lines, you obtain a line arrangement, this arrangement may not be prjective.

Problem 4.1: Construct line arrangements containing exactly $1, 2, 3$ and 4 points.

How many lines can they contain?

What happens if we ask for them to be projective line arrangements?

Solution 4.1:

We will care about some very special line arrangements.

A line arrangement is said to form a *combinatorial projective plane* if it furthermore satisfies the following properties:

- Given two points in $\mathcal{P}$, exactly one line in $\mathcal{L}$ passes through both of them.
- Given two lines in $\mathcal{L}$, exactly one point in $\mathcal{P}$ is contained in both of them.
- Every point in $\mathcal{P}$ is contained in the same number of lines in $\mathcal{L}$
- Every line in $\mathcal{L}$ contains the same number of points in $\mathcal{P}$.
- There are at least two points in $\mathcal{P}$.

Problem 4.2: Can you construct a combinatorial projective plane with exactly 3 points and 3 lines?

Can you construct a combinatorial projective plane with exactly 7 points and 7 lines?

What other line arrangements can be constructed if in the definition of combinatorial projective planes we allow for P to be only one point or $\mathcal{L}$ to be only one line.

Solution 4.2:

Let us focus on finite combinatorial projective planes. For a finite combinatorial projective plane, let

- N be the number of lines
- t be the number of points
- r be the number of points in a line
- k be the number of lines that contain a single point.

Problem 4.3: Show that $Nr = kt$.

Hint: Look at pairs (p, L), where $p \in L$.

Solution 4.3:

Problem 4.4: Show that $k(r-1) = t-1$

Hint: Fix a point and count the number of remaining points in two different ways.

Solution 4.4:

Problem 4.5: Show that $N = 1 + r(k-1)$.

Hint:Look at the points in one fixed line l_0, and regroup the other lines based on which of the points of l_0 they contain.

Solution 4.5:

Problem 4.6: Show that $r = k$ and $N = t$, i.e. there are the same number of points and lines.
Hint: Use the equalities from the previous Problems.

Solution 4.6:

Define $q := r - 1$, this value is called the order of a combinatorial projective plane.

Problem 4.7: Show that $N = r^2 - r + 1 = q^2 + q + 1$. Which agrees with the number of lines and points that we have computed for the projective planes over $\mathbb{F}_q$ in previous classes.

Solution 4.7:

You can use the following fact: Finite fields $\mathbb{F}_q$ can only have orders of the form $q = p^m$, where p is a prime number and n is a non-negative integer. For any number $q = p^n$, there exists a field of q elements.

Problem 4.8:

For which orders have we created combinatorial projective planes in the previous classes? (via the projective planes $\mathbb{P}^2_{\mathbb{F}_q}$)

Could there be combinatorial projective planes of any order?

Solution 4.8:

UCLA Mathematics Department, Los Angeles, CA 90095-1555, USA.
Email address: `fzamora@math.princeton.edu`

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA.
Email address: `jmoraga@math.ucla.edu`