Balanced Trees and Fibonacci numbers

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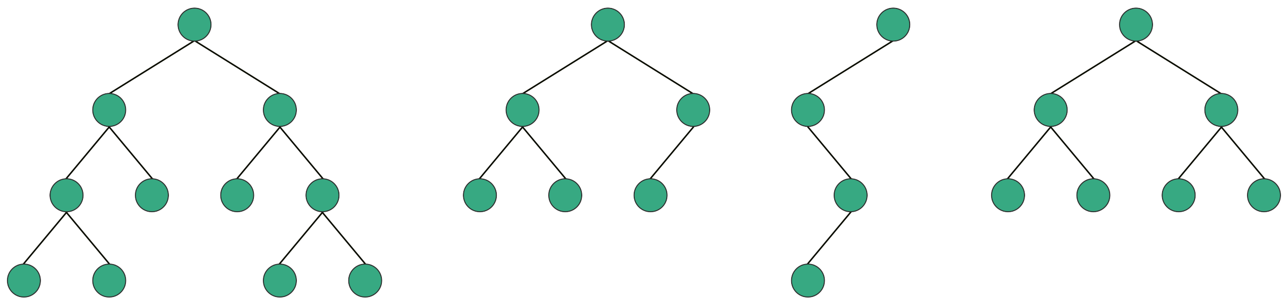
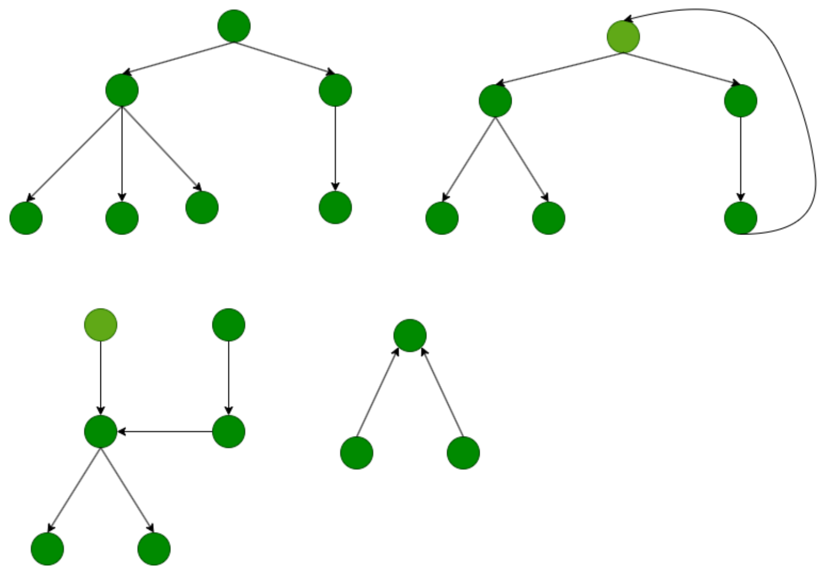
1 Binary Trees

Recall that we defined a **binary tree** as a tree where every node has at most two children and exactly one parent except the unique *root node* which does not have any parent. The left (or right) sub-tree of a node A in a tree is the binary tree stemming from A's left (or right) child. Here are some examples of binary trees:



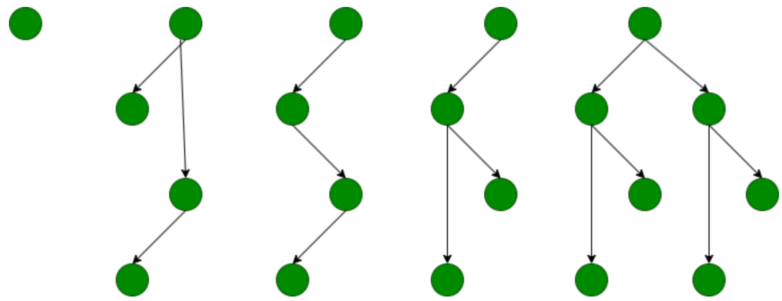
On the other hand, here are some graphs that are **not** binary trees:



Problem 1.1.
Identify which of the above graphs are trees.

We also defined the *height* of a binary tree informally as the number of levels it has. For notation, we say that a binary tree with only a root has height 0 and any other binary tree has height **equal to the largest number of downward steps one can take before you reach a leaf i.e. a node without any children.**

Problem 1.2.
Determine the height of each of the following binary trees.

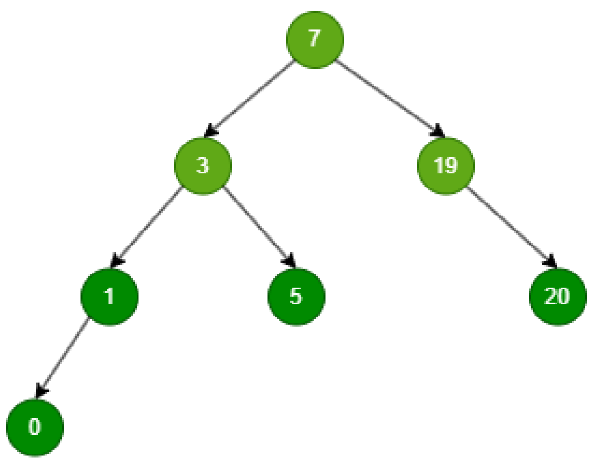


1.1 Warm-up with binary trees

Problem 1.3.
For a binary tree of height H, what is the maximum number of nodes N, it can have? What is then the minimum height of a tree with N nodes?

Problem 1.4.
What is the minimum number of nodes N of a binary tree with height H? What is then the maximum height of a tree given n nodes?

Suppose we assign **distinct** values to each of the nodes of a binary tree with height H, such that the value of each node is larger that all values on its left subtree and lower than those in its right subtree. We will call such a binary tree *valid*. For example the following is a valid binary tree:



Problem 1.5.
Create a valid binary tree that holds the numbers $1, 2, 3, 4$.

Problem 1.6.
Given 4 distinct numbers a_1, a_2, a_3, a_4 how many different valid trees can you create?

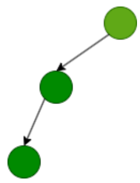
Problem 1.7.
Let $f(k)$ be the number of possible valid binary trees that hold k district values. Given $f(1), \cdots, f(N)$ find $f(N+1)$.
Hint: Think of the possible values for the root node, and the resulting left and right sub-trees it can have.

Problem 1.8.
Suppose you are given a valid binary tree with N nodes and height H. An evil computer scientist hid all the values in the nodes of the tree. He then gives you a number K and asks you to add a node to the tree with the value K, so that the tree remains valid. You are allowed to ask him to reveal some values of the tree, but you have to pay him a dollar for every node he reveals. In the worst case scenario, how much money would you have to give him to ensure that the node is placed correctly?

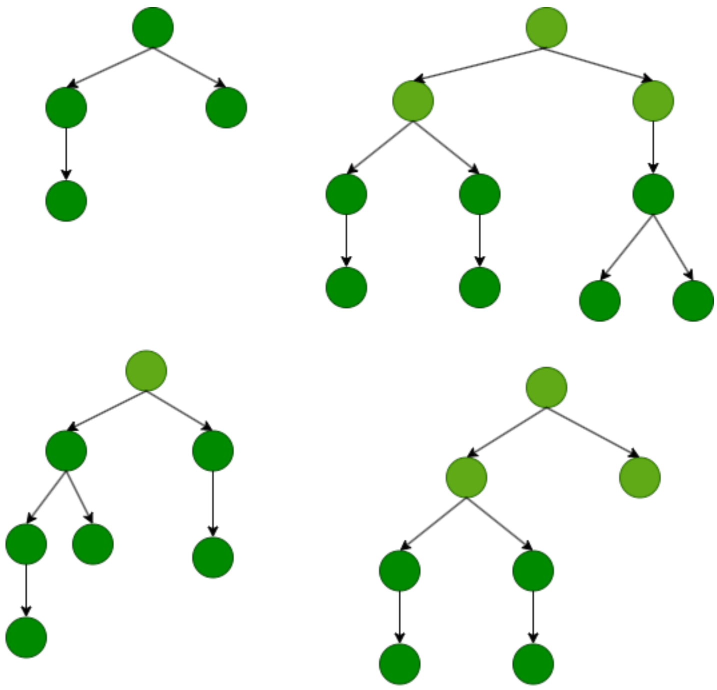
2 Balanced Trees

When it comes to binary trees, because of their purpose in computer science, the height of a tree can have a significant impact in the efficiency with which we perform operations such as adding an element to the tree *(Problem 1.8)*, or finding in which node a given value is. For such optimization purposes we introduce **Balanced** binary trees.

A balanced binary tree is one where for any given node the height of its right sub-tree is at most one different than that of its left sub-tree. As the height of a binary tree with one node is zero, to be consistent with calculations, we say that the "empty tree" has a height of -1.
So, the following is not a binary tree as the root's left sub-tree has height 1 and its left -1, for a difference of 2.



Problem 2.1.
Which of the following are balanced binary trees:



It should be clear that the condition that a binary tree is balanced somehow bounds its maximum height. We shall investigate the following question:

What is the minimum and maximum height of a balanced tree with N nodes?

Problem 2.2.
Given a balanced tree with N nodes what is the minimum height of a tree?

Finding an upper bound for a tree might be a bit more challenging. Instead lets ask an almost identical question: What is the minimum number of nodes on can fit in a tree of a given height H?

Problem 2.3.
Find the minimum amount of nodes you can place on a balanced binary tree of height H = -1, 0, 1, 2, and 3.

1. Looking at the case for $H = 4$, what should the height of the root's left and right sub-trees be?
2. To minimize the overall amount of nodes in the tree then, how many nodes does each of these sub-trees have.

Problem 2.4.
Let $min(H)$ denote the minimum number of nodes in a balanced binary tree of height H. Find $min(H)$ in terms of $min(H-1)$ and $min(H-2)$.

Recall the Fibonacci sequence: $F_0 = 0$,$F_1 = 1$, and for $n > 1, F_n = F_{n-1} + F_{n-2}$, resulting in the following:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots.$$

Problem 2.5.
Prove using induction and the answer to *Problem 2.4* that $min(H) = F_{H+3} - 1$

We have reduced the problem of finding the minimum amount of nodes in a balanced tree to finding a closed form solution to the Fibonacci sequence.

2.1 Finding F_n

To derive the formula for the n^{th} Fibonacci number we can ask the following question:

What is the ratio of two consecutive Fibonacci numbers: F_{n+1}/F_n as n gets very large?

Let $a_n = F_{n+1}/F_n$, the first few values of a_n are:

$$1,\ 2,\ 1.5,\ 1.666,\ 1.6,\ 1.625,\ 1.61538,\ 1.61904,\ 1.61764\ \cdots\ .$$

If one keeps listing these values they seem to be approaching a number close to 1.61803.

Problem 2.6.
Let a be the number that these ratios are approaching. Show that a would then have to satisfy $a = 1 + 1/a$

As the terms in the Fibonacci sequence get larger and larger it seems that the ratio between two consecutive terms approaches a constant. It is for that reason that in our attempt to find a formula for the n^{th} Fibonacci number we look at a sequence of the form: $\{x^n\}$. i.e. a sequence such that any two consecutive terms have a constant ratio of x.

To find the value of x that is of interest in this case we would have to impose on it the same condition that is imposed on the Fibonacci numbers, in this case: $x^n = x^{n-1} + x^{n-2}$.

Problem 2.7.
What are the two possible values of x that satisfy the equation above?

Problem 2.8.
Use induction to show then that for such x we have: $x^n = F_n \cdot x + F_n - 1$.

Problem 2.9.
Use the fact that both solutions to *Problem 2.7* satisfy the equation above, to finally derive a closed form formula for the n^{th} Fibonacci number.

Problem 2.10.
Now put everything together to find the minimum number of nodes in a balanced tree of height H.

Problem 2.11.
Bonus: Derive then the maximum height of a balanced tree with N nodes.

3 Fibonacci everywhere

It's incredible how observing a recursive relationship in balanced tree we ended up seeing the Fibonacci numbers!

3.1 Fun Problems

Problem 3.1.
In how many ways can you tile an $n \times 1$ block using only blocks of size 1×1 and 2×1.

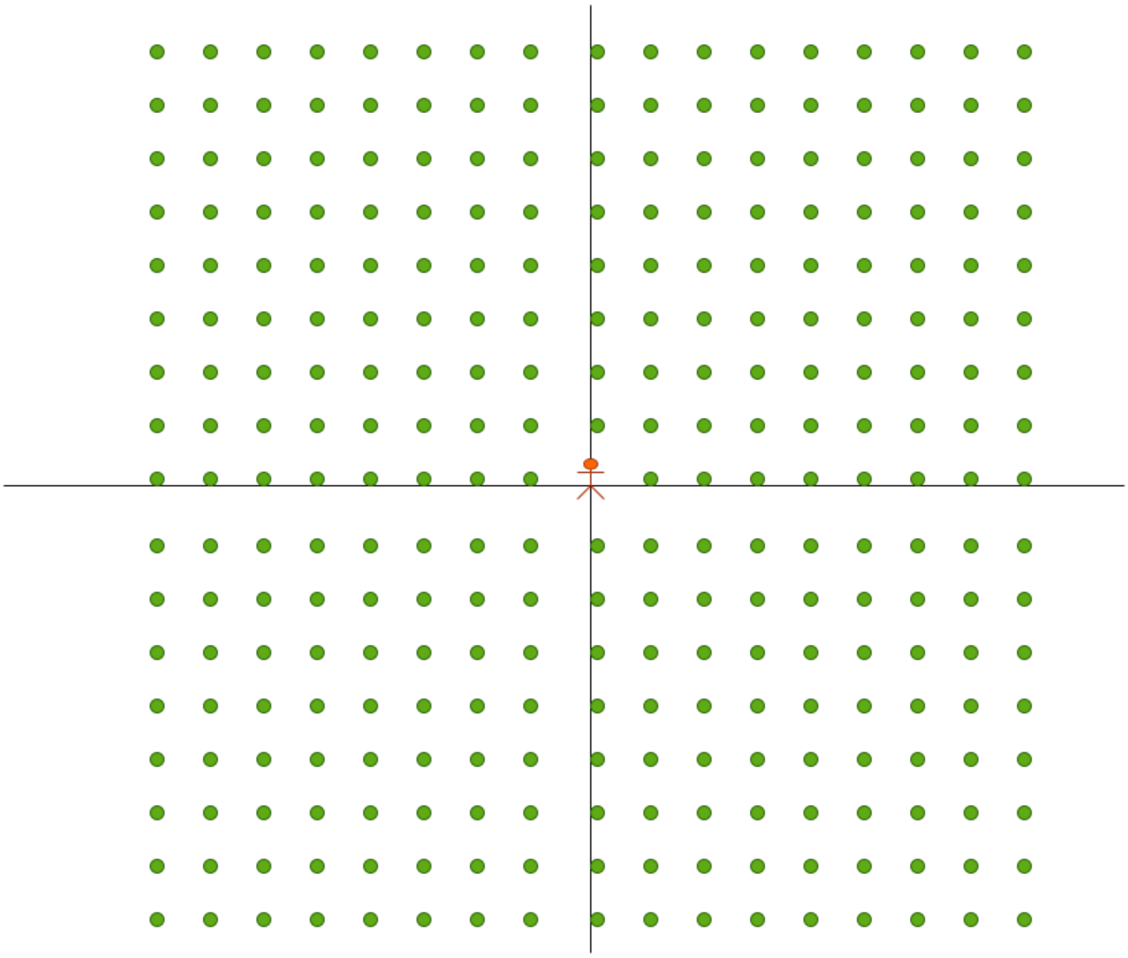
Problem 3.2.
Prove that given 7 straight sticks with lengths from 1 to 10 you can always find 3 with which you can form a triangle.

Problem 3.3.
Consider the sequence $a_n = 1000, x, 1000 - x, x - (1000 - x), \cdots$ where the $a_n = a_{n-2} - a_{n-1}$, suppose the sequence terminates when one of its terms becomes negative. What should the value of x be to maximize the length of the sequence?

3.2 ϕ is most irrational

Suppose you are standing in the middle of an infinite grid/field, where in every integer coordinate (i.e. (x, y) with x and y integers) there is a tree. An infinitely thin tree. In order to see the tree one has to look directly at it.

It is possible to look in a specific direction and never see a tree?



Problem 3.4.
Is is possible to ever see the tree with coordinates $(2, 4)$?

Problem 3.5.
What coordinates (x, y) must a tree have, so that its impossible to see?

Problem 3.6.
Let's say you look in a direction θ degrees above the positive x-axis and you see the tree with coordinates $(13, 27)$. What can you say about $tan(\theta)$?

Problem 3.7.
Suppose you look in a direction θ degrees above the positive x-axis such that $tan(\theta) = \sqrt{2}$. Are you ever going to see a tree?

Recall that we define irrational numbers the numbers that can't be expressed as a ratio of integers. One very famous example is $\pi \approx 3.14159265$.

Although π is irrational we can approximate it using rational numbers quite well: $22/7 - \pi \approx 0.0013$
$\pi - 333/106 \approx 0.000083$
$355/113 - \pi \approx 2.7 \cdot 10^{-7}$
But how does one obtain these approximations? We shall use π as an example:

Our first approximation will be just the integer part of π: $a_0 = 3$.
Our second approximation will be of the form $a_0 + \frac{1}{\lfloor a_1 \rfloor} = 3 + \frac{1}{\lfloor a_1 \rfloor}$. Solving for a_1 we get $a_1 = \frac{1}{\pi - 3}$, so $\lfloor a_1 \rfloor = 7$. So up to this point we have $\pi \approx 3 + \frac{1}{7}$
Note that since we used a flooring function on $\frac{1}{\pi-3}$ to get the integer part of a_1, in the next step we approximate a_1 in the same manner: $a_1 = \lfloor a_1 \rfloor + \frac{1}{\lfloor a_2 \rfloor} = 7 + \frac{1}{\lfloor a_2 \rfloor}$ Again we can solve for a_2 to get $\lfloor a_2 \rfloor = 15$. If we were to stop here our approximation would look like $\pi \approx 3 + \frac{1}{7 + \frac{1}{15}}$. If we continue in this manner then we are going to end up with

something of the form: $$a_0 + \frac{1}{\lfloor a_1 \rfloor + \frac{1}{\lfloor a_2 \rfloor + \frac{1}{\lfloor a_3 \rfloor + \frac{1}{\lfloor a_4 \rfloor + \cdots}}}}$$

Or in the case of π we get $$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \cdots}}}}$$

These representations of irrational numbers are called continued fractions. At any point one can choose to stop the approximation on some $\lfloor a_n \rfloor$, and retrieve the resulting rational number.
For example, if we were to stop with $\lfloor a_1 \rfloor$ in the case of π, we would get $\pi \approx 3 + 1/7$, which is $22/7$, the famous approximation for π.

Clearly, stopping at later terms in the sequence $\{a_n\}$ we get better rational approximations of our irrational number. We shall call an n^{th} degree approximation of r the rational number we get by terminating our continued fraction approximation at a_n (including a_n).

Problem 3.8.
Supppose you are standing in the middle of the grid and you are looking in a direction so that $tan(\theta)$ is exactly π. What would the thickness of the tree at (22,7) need to be so that you just miss it?

How can we get an irrational number with the worst possible rational approximations, and what would that mean for our infinite tree problem?

Problem 3.9.
Suppose you look once again from the middle of the grid outward with $tan(\theta) = q$. Suppose q is such that you almost saw the tree at (355/113). Can you find a good rational approximation for q?

Problem 3.10.
Take two irrational numbers q_1, q_2 and consider their 1^{st} degree approximations: $q_1 \approx 1 + \frac{1}{a}$

and $q_2 \approx 1+\frac{1}{b}$. If $a > b$, which number has a better 0^{th} degree approximation (i.e. integer approximation).

Problem 3.11.
If we wanted to make the 0^{th} degree approximation of q_2 as "bad" as possible what value would we have to choose for b?

Problem 3.12.
Take two irrational numbers w_1, w_2 and consider their 2^{nd} degree approximations: $w_1 \approx 1+\frac{1}{1+\frac{1}{a}}$ and $w_2 \approx 1+\frac{1}{b}$. If $a > b$, which number has a better 1^{st} degree approximation?

Problem 3.13.
If we wanted to make the 1^{st} degree approximation of w_2 as "bad" as possible what value would we have to choose for b?

You might be noticing a pattern arise at this point. It seems that the larger $\lfloor a_{n+1} \rfloor$ is the better the n^{th} rational approximation of our irrational number is.

In light of the realization above, how could one choose $\{a_1, a_2, a_3, \cdots\}$ in the continued fraction representation to create an irrational number with the worst possible rational approximations?

Clearly, making $\{a_1, a_2, a_3, \cdots\} = \{1, 1, 1, 1, \cdots\}$ would achieve the desired result as a_n are all integers and the smallest value they can have is 1. Let's call the resulting number: z.

Now suppose we looked outward with an angle of θ such that $tan(\theta) = z$. Notice that our number that has the worst possible rational approximation will be also the one furthest away from seeing any tree in the way. Since being close to seeing a tree is analagous to having a good rational approximation.

Problem 3.14.
Based on the expression $z = 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}}$. Can you find the value of z? *Maybe look at some of its small degree approximations to get an idea.*