# INEQUALITIES

MATH CIRCLE (ADVANCED) 10/28/2012

1) Which number is greater

a) $31^{11}$ or $17^{14}$?

$31^{11} < 32^{11} = 2^{55} < 2^{56} = 16^{14} < 17^{14}$

b) $2^{300}$ or $3^{200}$?

$2^{300} = 8^{100} < 9^{100} = 3^{200}$

c) $2^{40}$ or $3^{28}$?

$2^{40} = 1024^4 < 2187^4 = 3^{28}$

d) $100^{100}$ or $150^{50} \cdot 50^{50}$?

$100^{100} = (100^2)^{50} \geq (150 \cdot 50)^{50} = 150^{50}50^{50}$

2) a) Prove that $2^{100} + 3^{100} < 4^{100}$.

$2^{100} + 3^{100} < 2 \cdot 3^{100}$. Then note $(\frac{4}{3})^{100} > (\frac{4}{3})^3 > 2$.

b) Prove that

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{98} - \frac{1}{99} + \frac{1}{100} > \frac{1}{5}.$$

Hint: Group the terms to get a sum of positive numbers. Then even $\frac{1}{6} + \frac{1}{20} > \frac{1}{5}$.

3) Prove the following:

a) If $a \geq b$ and $x \geq y$, then $ax + by \geq ay + bx$.

$ax + by - ay - bx = (a - b)(x - y) \geq 0$.

b) $\frac{a^2}{4} + b^2 + c^2 \geq ab - ac + 2bc$ for all $a, b, c$.

$\frac{a^2}{4} + b^2 + c^2 - ab + ac - 2bc = (\frac{a}{2} - b + c)^2 \geq 0$

c) If $a + b + c = 0$, then $ab + bc + ca \leq 0$.

$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = -(a^2 + b^2 + c^2) \leq 0$

4) Which number is greater:

a) $1234567 \cdot 1234569$ or $1234568^2$?

In general, $(x - 1)(x + 1) = x^2 - 1 < x^2$

b) $1234567/7654321$ or $1234568/7654322$?

Hint:

$$\frac{x}{y} - \frac{x+1}{y+1} = \frac{x-y}{y(y+1)}$$

5) We are given the two fractions

$$\frac{10\ldots01}{10\ldots001} \text{ and } \frac{100\ldots01}{100\ldots001}$$

where each fraction has one more zero in the denominator than in the numerator. If the numerator in the left fraction has $1984$ zeros, and the numerator in the right fraction has $1985$ zeros, which of them is greater?

Hint: Let $x$ stand for the numerator, then the whole fraction $a$ satisfies

$$a = \frac{x}{10x - 9}.$$

What happens to $a$ as $x$ increases?

6)* Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{99}{100} < \frac{1}{10}.$$

If $A = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{99}{100}$ and $B = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{98}{99}$ then $A \cdot B = \frac{1}{100}$. Then note $A < B$.

7)* Prove that if $a_1 \leq a_2 \leq a_3 \leq a_4$ and $b_1 \leq b_2 \leq b_3 \leq b_4$ then

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \geq a_1c_1 + a_2c_2 + a_3c_3 + a_4c_4,$$

where $c_1, c_2, c_3, c_4$ is an arbitrary permutation of the numbers $b_1, b_2, b_3, b_4$.

Hint: Use 3a) on each transposition.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg “Mathematical Circles (Russian Experience)”