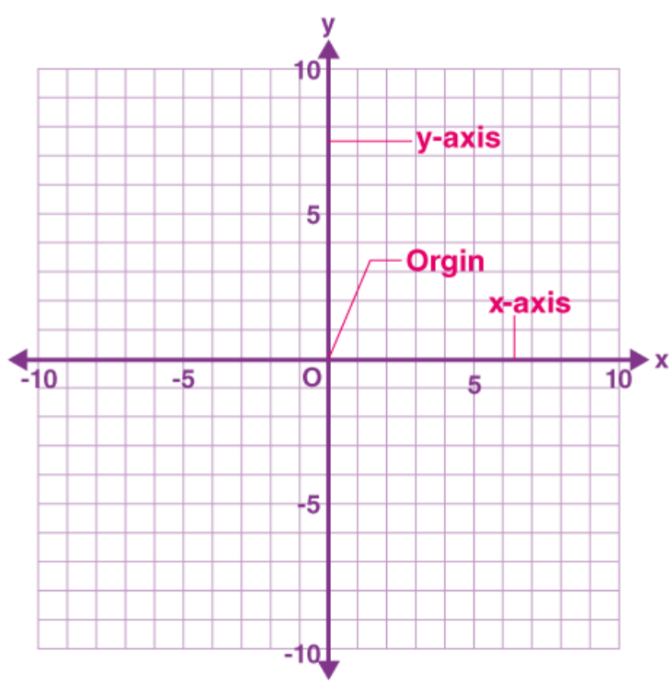
**Intermediate 1** **Jan 21, 2024**

# Taxicab Geometry

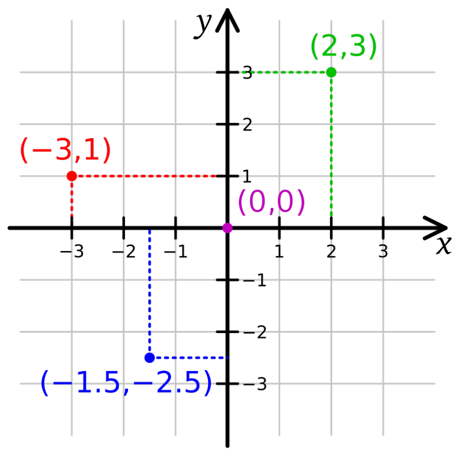
## 1 Cartesian Plane

Coordinate plane is a convenient way to describe planar geometry. Coordinates are numbers we can use to tell exactly where the points are located.

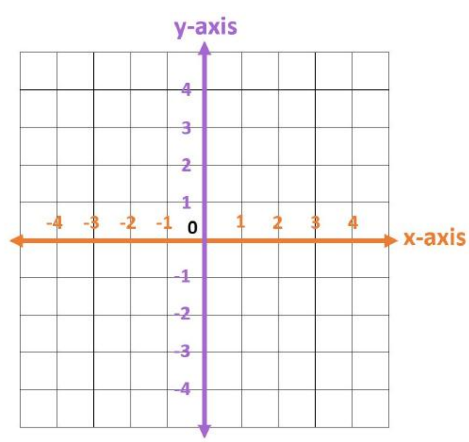
Recall from last quarter that the **Cartesian plane** is a plane with perpendicular lines. The horizontal line is known as $X$-axis, and the vertical line is known as $Y$-axis. The intersection of the $X$- and $Y$- axis is called the origin.



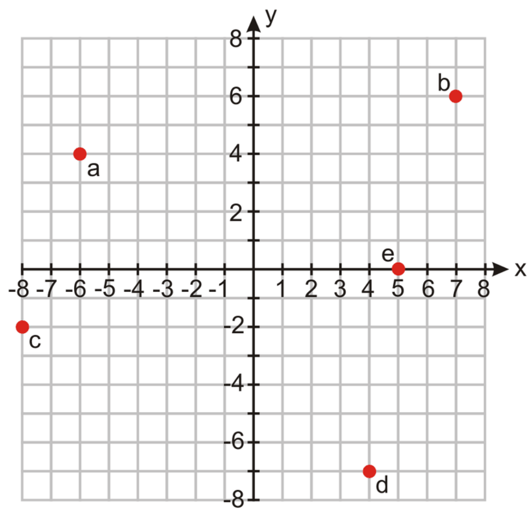
On the Cartesian plane, every point can be represented by a pair of real numbers $(x, y)$, called the **coordinate**, where $x$ indicates the horizontal distance of the point from the origin, and $y$ represents the vertical distance of the point from the origin. If the sign of $x$ is positive, the point is on the right of the origin; else it is on the left. Similarly, if the sign is positive for y, the point is y points above the origin else it is below it.




**Problem 1.1.** *Draw the points* $(0, 3), (4, 0)$, *and* $(-2, -3)$ *below.*



**Problem 1.2.** *Label the points $a, b, c, d, e$ below with their coordinates.*

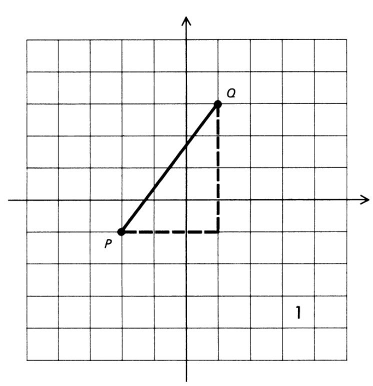



In Euclidean geometry, the Euclidean distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is defined as

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We denote the Euclidean distance between $P$ and $Q$ as $d_E(P, Q)$.

For example, if the point $P = (-2, -1)$ and $Q = (1, 3)$,



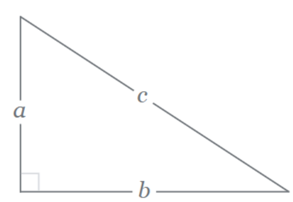
Then the distance between $P$ and $Q$ is

$$d_E(P,Q) = \sqrt{(1-(-1))^2 + (3-(-1))^2} = 5.$$

This definition of distance originated from the Pythagorean theorem in Euclidean geometry. One might prove this theorem in the traditional Euclidean geometry setting, without referring to the definition of distance above. (Try to prove this yourself using what we learned about similar triangles!)

**Theorem 1.** *Let $\triangle ABC$ be a right triangle in Euclidean geometry. Let $a, b$ be the length of the two legs, and $c$ be the length of the hypotnuse. Then*

$$a^2 + b^2 = c^2.$$

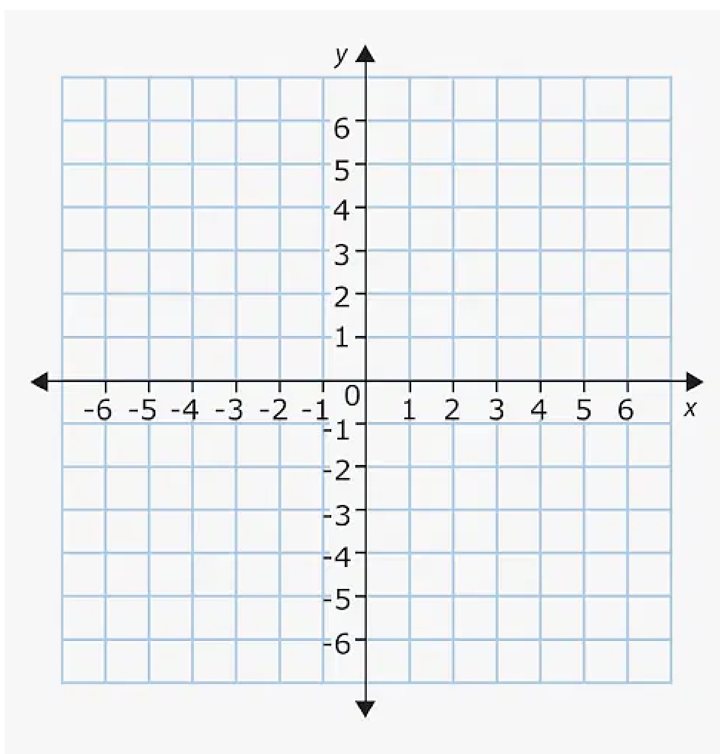



**Problem 1.3.** *Find the Euclidean distance between the points $(0,3)$ and $(3,0)$.*

**Problem 1.4.** *Bob walks starting from his house 60 feet to the east and 80 feet to the north. How far away he is from his house?*

**Problem 1.5.** *In the Cartesian plane below, let $P$ be the point $(2, 2)$. graph the figure containing all points $x$ that satisfies*

$$d_E(x, P) = 3$$



## 2 Taxicab Geometry

One way to describe a geometry is to tell what its points are, what is a straight line, how distance is measured, and how angle measured are determined.

So far the geometry we are working with is exclusively the Euclidean geometry, where the points are objects that cannot be divided into parts; straight lines are long, thin, evenly spread sets of points; angles are measured in degrees by a perfect protractor, and distance is given by the Euclidean distance $d_E$ mentioned above.

Now we introduce a new kind of geometry: the **taxicab geometry**. In this new geometric world, points, lines, and angles are exactly the same as in Euclidean geometry. The only difference is the way to calculate distance between two points.

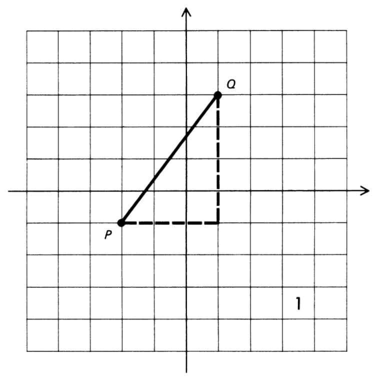
**Definition 1.** *Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. The **taxicab distance** between $P$ and $Q$ is defined as*

$$|x_2 - x_1| + |y_2 - y_1|$$

*denoted as $d_T(P, Q)$.*

The idea of taxicab distance is the following: imagine we are living in a city (such as Manhattan) where all streets go either in the horizontal direction or the vertical direction. If we want to go from one place to another, the distance we need to travel is the horizontal distance we need to go plus the vertical distance we need to go.

For example, if the point $P = (-2, -1)$ and $Q = (1, 3)$,

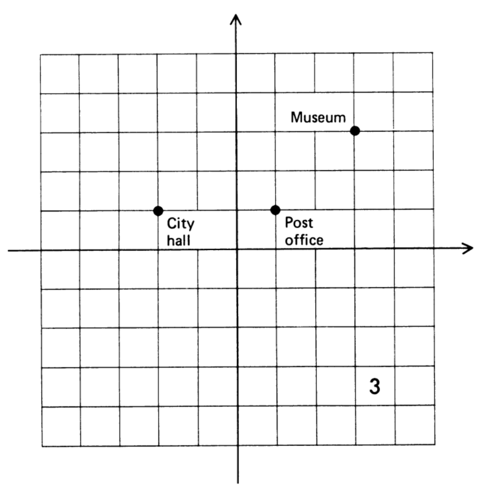



Then the taxicab distance between $P$ and $Q$ is

$$d_T(P, Q) = |1 - (-2)| + |3 - (-1)| = 3 + 4 = 7$$

because to go from $P$ to $Q$, we need to go 3 units to the right and 4 units upward.

**Problem 2.1.** *Assuming a taxi only goes in horizontal or vertical direction. What is the minimum distance needed to travel from the city hall to the museum?*



**Problem 2.2.** *In the same setting as in the previous problem, which place is farther away from the post office, museum or city hall?*

*Would the answer to this question be different if we are using Euclidean geometry?*

**Problem 2.3.** *Find the taxicab distance* $d_T(P, Q)$ *for all* $P$ *and* $Q$ *below:*

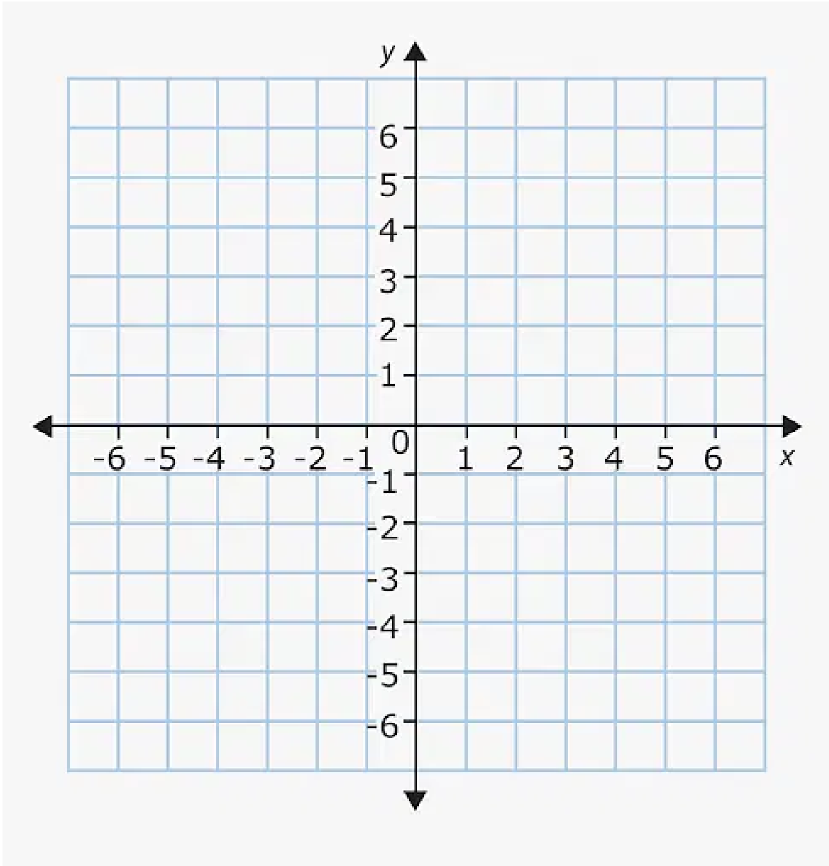
- $P = (5, 4)$ *and* $Q = (1, 2)$.

- $P = (-4, 3)$ *and* $Q = (3, 2)$.

- $P = (-5, -4)$ *and* $Q = (1, -2)$.

- $P = (3, -1)$ *and* $Q = (-2, 4)$.

**Problem 2.4.** *In the Cartesian plane below, let $P$ be the point $(2, 2)$.*



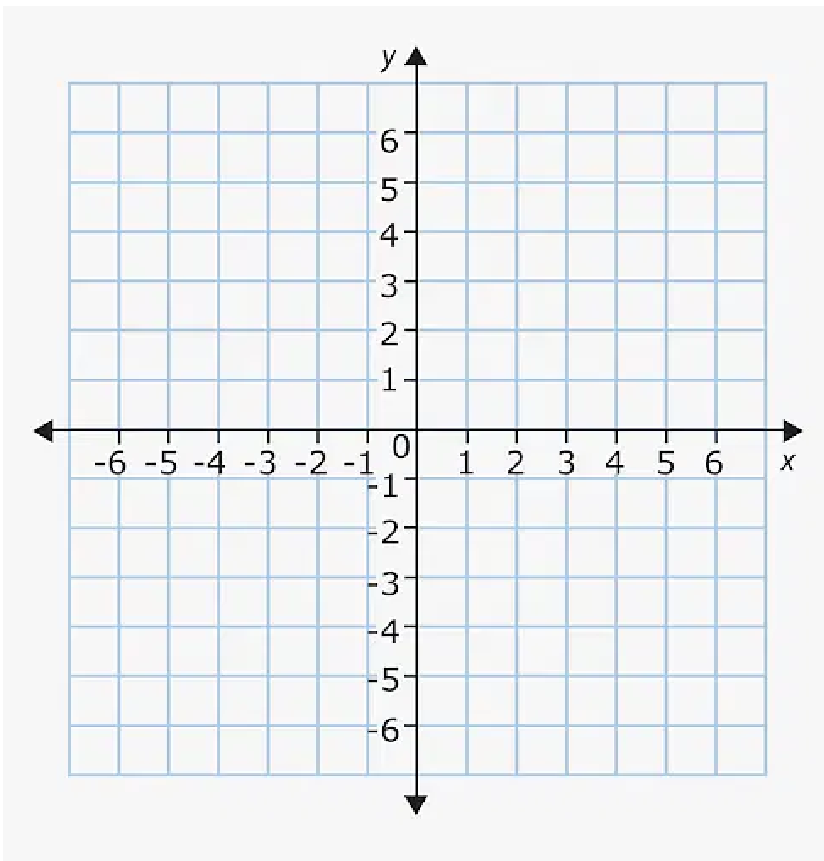

- *Draw some points that has a taxicab distance 3 away from $P$.*
- *Draw a figure containing all points with taxicab distance 3 from $P$. That is, graph the set*

$$\{x \in \mathbb{R}^2 \ : \ d_T(x, P) = 3\}.$$

- *Invent a reasonable name for the figure you've drawn.*

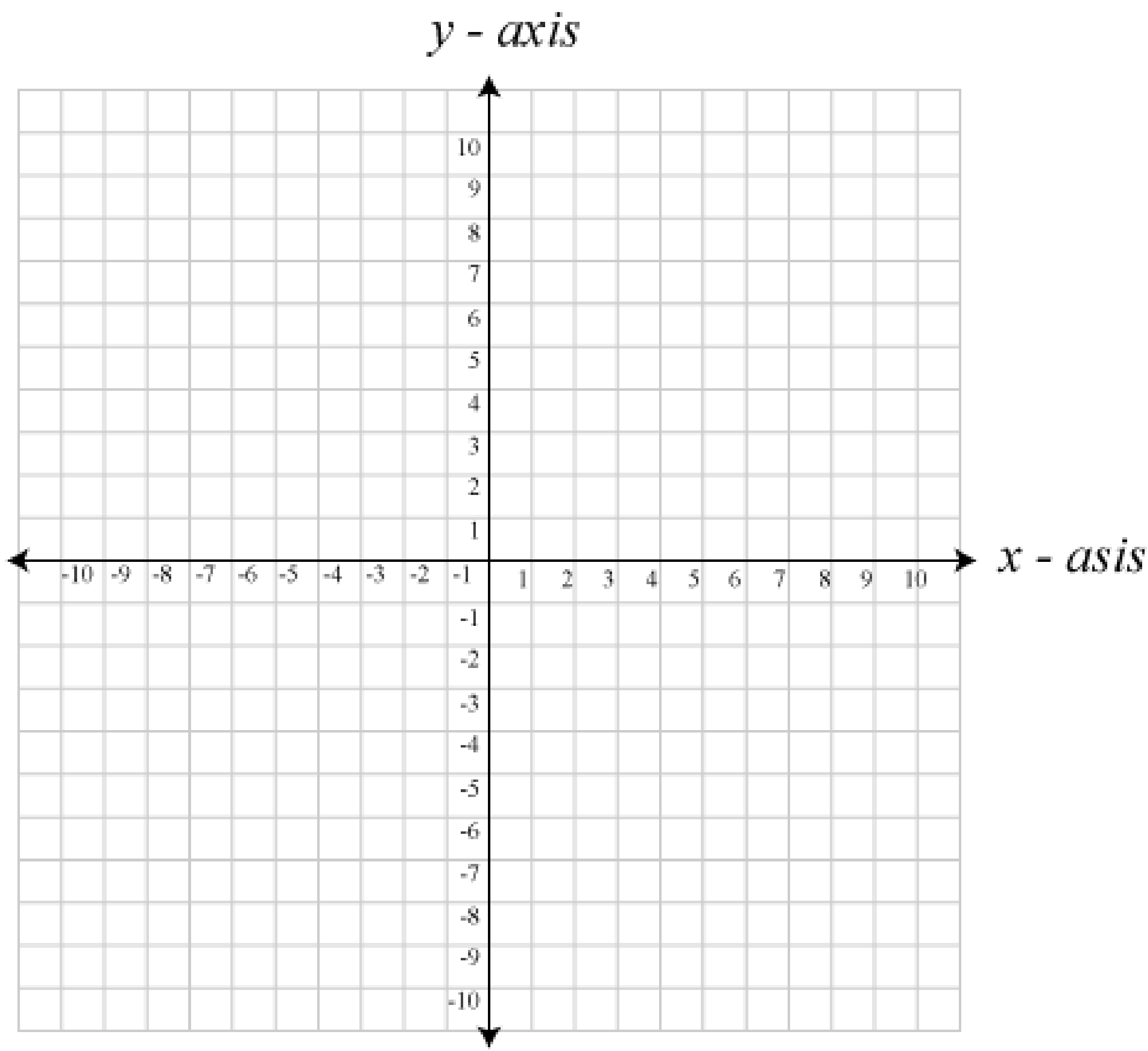
**Problem 2.5.** *$\pi$ is the ratio of a circle's circumference to its diameter. What is the numerical value of $\pi$ in taxicab geometry?*

**Problem 2.6.** *In the Cartesian plane below, draw a taxicab circle with center $(-2, -1)$ and radius 2. Then draw a taxicab circle with center $(3, 2)$ and radius 2.5.*



**Problem 2.7.** *Given* $A = (-2, -1)$ *and* $B = (3, 2)$.

- *Calculate* $d_T(A, B)$.

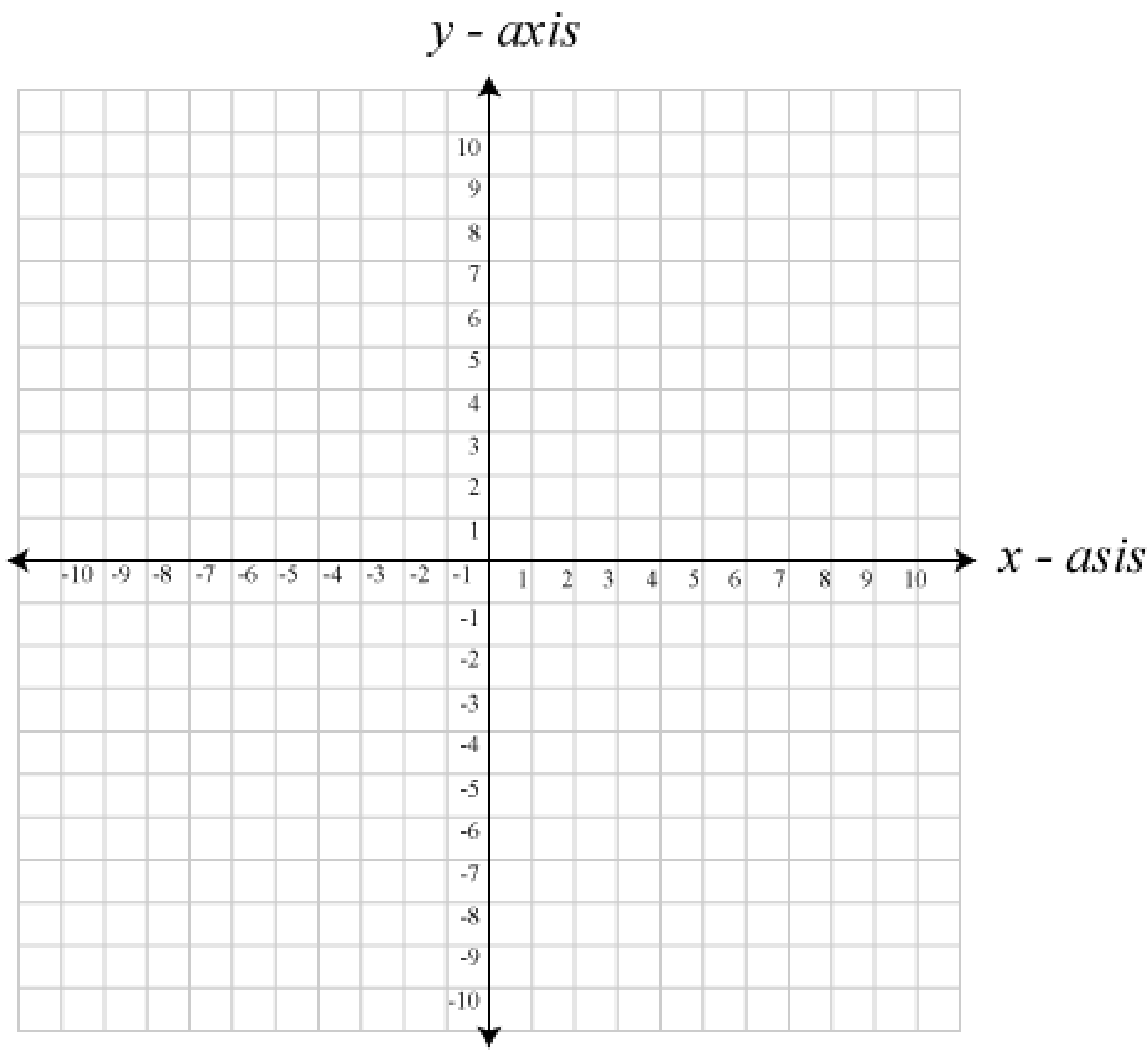


*Now on the previous Cartesian plane, graph the following sets:*

- $\{P|d_T(P,A) = 3 \ and \ d_T(P,B) = 5\}$
- $\{P|d_T(P,A) = 1 \ and \ d_T(P,B) = 7\}$
- $\{P|d_T(P,A) = 0 \ and \ d_T(P,B) = 8\}$
- $\{P|d_T(P,A) = 1.5 \ and \ d_T(P,B) = 6.5\}$
- $\{P|d_T(P,A) = 4 \ and \ d_T(P,B) = 4\}$
- $\{P|d_T(P,A) = 5 \ and \ d_T(P,B) = 3\}$
- $\{P|d_T(P,A) + d_T(P,B) = d_T(A,B)\}$

**Problem 2.8.** *Given $A = (-2, -1)$ and $B = (3, 2)$.*

- *Calculate $d_T(A, B)$.*



*Now on the previous Cartesian plane, graph the following sets:*

- $\{P|d_T(P, A) = 5 \ and \ d_T(P, B) = 5\}$

- $\{P|d_T(P, A) = 7 \ and \ d_T(P, B) = 7\}$

- $\{P|d_T(P, A) = 4 \ and \ d_T(P, B) = 4\}$

- $\{P|d_T(P, A) + d_T(P, B) = d_T(A, B)\}$

**Problem 2.9.** *Repeat the previous exercise using $d_E$ in place of $d_T$. (A compass will be helpful.)*