ORMC AMC 10/12 Group
Winter, Week 4: Number Theory

Jan 28, 2024

1 Warm-up Exercises

1. **(2017 AMC 8 #24)** The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

2. **(1992 AHSME #23)** Let S be a subset of $\{1, 2, 3, ..., 50\}$ such that no pair of distinct elements in S has a sum divisible by 7. What is the maximum number of elements in S?

3. **(2000 AMC 12 #18)** In year N, the 300^{th} day of the year is a Tuesday. In year $N+1$, the 200^{th} day is also a Tuesday. On what day of the week did the 100th day of year $N-1$ occur?

4. **(1997 AIME #1)** How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?

5. **(1987 AIME #5)** Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

2 Number Theory Topics

One of the most important facts when working with integers the **Fundamental Theorem of Arithmetic**, which states that **every integer has a unique prime factorization**. The proof is more advanced, so it won't be covered here. The important takeaway from this theorem is that since primes generally operate "orthogonally" (similar to a *vector basis*, in linear algebra), we can often formulate conclusions for integers by figuring something out for powers of primes, and then generalizing it to all other numbers.

For example, given two integers a, b, consider the **Greatest Common Divisor** $\gcd(a, b)$, the largest number which divides both a and b, and the **Least Common Multiple** $\text{lcm}(a, b)$, the smallest number divisible by both a and b. If a and b are powers of primes, then there are two cases:

1. If $a = p^m$ and $b = p^n$, then $\gcd(a, b) = p^{\min(m,n)}$ and $\text{lcm}(a, b) = p^{\max(m,n)}$.
2. If $a = p^m$ and $b = q^n$ where p, q are distinct primes, then $\gcd(a, b) = 1$ and $\text{lcm}(a, b) = ab$.

So, in general, we have the following:

$$\text{If } a = p_1^{m_1} \cdots p_k^{m_k} \text{ and } b = p_1^{n_1} \cdots p_k^{n_k},$$
$$\text{then } \boxed{\gcd(a, b) = p_1^{\min(m_1,n_1)} \cdots p_k^{\min(m_k,n_k)}} \text{ and } \boxed{\text{lcm}(a, b) = p_1^{\max(m_1,n_1)} \cdots p_k^{\max(m_k,n_k)}}.$$

In particular, notice that this means $\text{lcm}(a, b) \cdot \gcd(a, b) = a \cdot b$.

But sometimes, finding the prime factorization of a number isn't very easy, so we want a better way to find the gcd. We can use the **Euclidean Algorithm**, which is based on the fact that if d divides both a and b, then we can write $a = dm, b = dn$, and then $a - b$ is divisible by d, since it is equal to $dm - dn = d(m - n)$. In particular, $\gcd(a, b) = \gcd(b, a(\text{mod } b))$, where $a(\text{mod } b)$ is shorthand for "the remainder when a is divided by b." If we iterate this process of finding remainders, we will eventually find the gcd.

For example:

$$\begin{aligned}
\gcd(67620, 66234) &= \gcd(66234, 67620(\text{mod } 66234)) = \gcd(66234, 1386) = \gcd(1386, 66234(\text{mod } 1386)) \\
&= \gcd(1386, 1092) = \gcd(1092, 1386(\text{mod } 1092)) \\
&= \gcd(1092, 294) = \gcd(294, 1092(\text{mod } 294)) \\
&= \gcd(294, 210) = \gcd(210, 294(\text{mod } 210)) \\
&= \gcd(210, 84) = \gcd(84, 210(\text{mod } 84)) \\
&= \gcd(84, 42) = \gcd(42, 84(\text{mod } 42)) = \gcd(42, 0) = \boxed{42}.
\end{aligned}$$

One final important topic in number theory for the AMC 10/12 is the **Chinese Remainder Theorem**, which allows you to solve systems of congruences, as well as finding remainders modulo composite numbers. The theorem states that if $m_1, \ldots, m_k$ are pairwise relatively prime, then the system of congruences

$$x \equiv a_1 \pmod{m_1}, \ldots, x \equiv a_k \pmod{m_k}$$

has a unique solution mod $m_1 \cdots m_k$ (i.e., there is a unique integer solution in $1, 2, \ldots, m_1 \cdots m_k$).

The theorem only tells us that a solution *exists*, but not how to find it. Here's an example of how to solve this type of system of congruences: Find the smallest positive value of n such that $n \equiv 4 \pmod 5, n \equiv 3 \pmod 6$, and $n \equiv 2 \pmod 7$.

- Start with the largest modulus, 7. Since $n \equiv 2 \pmod 7$, we can write $n = 7m + 2$, for some integer m.
- Since $n \equiv 3 \pmod 6$, we know $7m + 2 \equiv 3 \pmod 6$, so $m \equiv 1 \pmod 6$. Now, we can write $m = 6k + 1$, so $n = 7(6k + 1) + 2 = 42k + 9$.
- Plugging into the final congruence $n \equiv 4 \pmod 5$, we find that $42k + 9 \equiv 4 \pmod 5$, so $2k \equiv 0 \pmod 5$. So $k = 5l$ for some integer l, and $n = 210l + 9$ for some integer l.
- Since $5, 6, 7$ are pairwise relatively prime and $5 \cdot 6 \cdot 7 = 210$, the chinese remainder theorem tells us that $n \equiv 9 \pmod{210}$ is the unique solution for n, modulo 210, so the smallest positive value of n is $\boxed{9}$.

3 Exercises

1. **(2005 AIME I # 2)**For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, 10, \ldots$. For how many values of k does S_k contain the term 2005?

2. **(2000 AIME I #1)** Find the least positive integer n such that no matter how 10^n is expressed as the product of any two positive integers, at least one of these two integers contains the digit 0.

3. **(1986 AHSME #23)** Let $\mathrm{N} = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1$. How many positive integers are factors of N?

4. **(1959 IMO #1)** Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

5. **(2000 AIME II #4)** What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?

6. Mr. Yu wants to divide the class into groups. When he tries to divide into groups of 3, there is 1 student left over. When he tries groups of 4, 1 student is left over. And when he tries groups of 5, there is still 1 student left over. What is the smallest number of students he could have, assuming he has more than 1 student?

7. **(2020 AMC 8 #17)** How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6$, and 12.)

8. **(1960 AHSME #33)** You are given a sequence of 58 terms; each term has the form $P + n$ where P stands for the product $2 \times 3 \times 5 \times \ldots \times 61$ of all prime numbers less than or equal to 61, and n takes, successively, the values $2, 3, 4, \ldots, 59$. Let N be the number of primes appearing in this sequence. Then N is:

9. **(1969 AHSME #23)** For any integer $n > 1$, how many prime numbers are greater than $n! + 1$ and less than $n! + n$?

10. **(1998 AIME #1)** For how many values of k is 12^{12} the least common multiple of the positive integers 6^6, 8^8, and k?

11. **(2018 AMC 10B #23)** How many ordered pairs (a, b) of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot \mathrm{lcm}(a, b) + 12 \cdot \gcd(a, b),$$

where $\gcd(a, b)$ denotes the greatest common divisor of a and b, and $\mathrm{lcm}(a, b)$ denotes their least common multiple?

12. **(2017 AMC 12B #19)** Let $N = 123456789101112\ldots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

13. **(1986 AIME #5)** What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

14. **(1970 AHSME #34)** What is the greatest integer that leaves the same remainder when dividing each of 13511, 13903 and 14589?

15. **(1985 AIME #13)** The numbers in the sequence 101, 104, 109, 116,... are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$ For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

16. **(1990 AIME #5)** Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $\frac{n}{75}$.

17. **(2010 AMC 12A #23)** The number obtained from the last two nonzero digits of $90!$ is equal to n. What is n?

18. **(2020 AMC 10A #24)** Let n be the least positive integer greater than 1000 for which

$$\gcd(63, n + 120) = 21 \quad \text{and} \quad \gcd(n + 63, 120) = 60.$$

What is the sum of the digits of n?

19. **(2021 AIME II #9)** Find the number of ordered pairs (m, n) such that m and n are positive integers in the set $\{1, 2, ..., 30\}$ and the greatest common divisor of $2^m + 1$ and $2^n - 1$ is not 1.

20. **(2021 AIME I #10)** Consider the sequence $(a_k)_{k\geq 1}$ of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n, then

$$a_{k+1} = \frac{m + 18}{n + 19}.$$

Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\frac{t}{t+1}$ for some positive integer t.