OLGA RADKO MATH CIRCLE, WINTER 2024: ADVANCED 3

FERNANDO FIGUEROA AND JOAQUÍN MORAGA

Worksheet 3:

Throughout this worksheet $\mathbb{F}$ is a field.

Remember that the *Projective Space* $\mathbb{P}^n_{\mathbb{F}}$ is defined to be the set of points with coordinates $[x_0 : x_1 : \ldots : x_n]$, with $x_i \in \mathbb{F}$ not all of them equal to 0, where two sets of coordinates $[x_0 : \ldots : x_n]$ and $[y_0 : \ldots y_n]$ define the same point if there exists a constant $\lambda \in \mathbb{F}$, such that for all i:

$$x_i = \lambda y_i.$$

And a hyperplane in projective space $\mathbb{P}^n_{\mathbb{F}}$ is the zero-set of a degree one polynomial, i.e. it is the elements $[x_0 : \ldots : x_n]$ satisfying the equation:

$$a_0 x_0 + \ldots + a_n x_n = 0,$$

For some fixed $a_i \in \mathbb{F}$.

Problem 3.1: Count how many points there are in the following hyperplanes

- $x_0 = 0$ in $\mathbb{P}^3_{\mathbb{F}_5}$
- $x_0 - x_1 + 2x_2 = 0$ in $\mathbb{P}^4_{\mathbb{F}_2}$
- $3x_0 + x_1 = 0$ in $\mathbb{P}^2_{\mathbb{F}_{11}}$
- Any hyperplane in $\mathbb{P}^n_{\mathbb{F}_q}$

Solution 3.1:

A line in projective space $\mathbb{P}^n_{\mathbb{F}}$ can be defined by a linear projective parametrization with parameters r, s, i.e. the points in a line are those that have coordinates

$$[p_0(r, s) : \ldots : p_n(r, s)]$$

where p_i are homogeneous degree 1 polynomials over $\mathbb{F}$, and r, s take values in $\mathbb{F}$ other than $r = s = 0$, such that the polynomials do not all vanish at the same time.

Alternatively, a line in the projective plane $\mathbb{P}^2_{\mathbb{F}}$ can also be defined as the zero-set of a degree one polynomial, i.e. it is the elements $[x_0 : x_1 : x_2]$ satisfying the equation:

$$a_0x_0 + a_1x_1 + a_2x_2 = 0,$$

For some fixed $a_i \in \mathbb{F}$.

Problem 3.2: What is the minimum size of a set of points $P \subseteq \mathbb{P}^2_{\mathbb{F}_2}$, such that every line in $\mathbb{P}^2_{\mathbb{F}_2}$ passes through at least one point in P?

Solution 3.2:

Problem 3.3: What is the minimum size of a set of lines $L \subseteq \mathbb{P}^2_{\mathbb{F}_3}$, such that every point in $\mathbb{P}^2_{\mathbb{F}_3}$ is contained in at least one line in L?
Solution 3.3:

Let $\mathbb{P}^2_{\mathbb{F}}$ have coordinates $[x_0 : x_1 : x_2]$

The dual projective plane $\mathbb{P}^{2\vee}_{\mathbb{F}}$ is defined as the set of points with coordinates $[a_0 : a_1 : a_2]$, with $a_i \in \mathbb{F}$ not all of them equal to 0, where two sets of coordinates $[a_0 : a_1 : a_2]$ and $[b_0 : b_1 : b_2]$ define the same point if there exists a constant $\lambda \in \mathbb{F}$, such that for all i:

$$a_i = \lambda b_i.$$

Here, points in $\mathbb{P}^{2\vee}_{\mathbb{F}}$ are in bijection with lines in $\mathbb{P}^2_{\mathbb{F}}$ via the following map: To a point $[a_0 : a_1 : a_2]$ in $\mathbb{P}^{2\vee}_{\mathbb{F}}$, we associate the line $a_0x_0 + a_1x_1 + a_2x_2 = 0$ in $\mathbb{P}^2_{\mathbb{F}}$.

In this way the dual projective plane is a projective plane whose points correspond to lines in the projective plane.

Problem 3.4: Establish a bijection from lines in $\mathbb{P}^{2\vee}_{\mathbb{F}}$ to points in $\mathbb{P}^2_{\mathbb{F}}$.

The lines in the dual projective plane $\mathbb{P}^{2\vee}_{\mathbb{F}}$, will be the zero-sets of degree one polynomials, i.e. the elements $[a_0 : a_1 : a_2]$ satisfying the equation:

$$x_0a_0 + x_1a_1 + x_2a_2 = 0,$$

For some fixed $x_i \in \mathbb{F}$.

Solution 3.4:

Let p be the intersection of two lines l_1 and l_2 in $\mathbb{P}^2_{\mathbb{F}}$. Let $p^\vee$, $l_1^\vee$ and $l_2^\vee$ be the corresponding line and points in $\mathbb{P}^{2\vee}_{\mathbb{F}}$, via the bijections from the previous problem.

Problem 3.5: Show that the line $p^\vee$ contains the points $l_1^\vee$ and $l_2^\vee$.

Solution 3.5:

Problem 3.6: Using the Problems 3.4 and 3.5, explain how Problems 3.2 and 3.3 are the same problem for different fields.
Solution 3.6:

Problem 3.7: What is the minimum size of a set of points $P \subseteq \mathbb{P}^2_{\mathbb{F}_p}$, such that every line in $\mathbb{P}^2_{\mathbb{F}_p}$ passes through at least one point in P?

Solution 3.7:

Problem 3.8: What is the minimum size of a set of points $P \subseteq \mathbb{P}^3_{\mathbb{F}_2}$, such that every hyperplane in $\mathbb{P}^3_{\mathbb{F}_2}$ passes through at least one point in P?

What about for points and hyperplanes in $\mathbb{P}^3_{\mathbb{F}_q}$?

Solution 3.8:

Problem 3.9: What is the minimum size of a set of points $P \subseteq \mathbb{P}^n_{\mathbb{F}_q}$, such that every hyperplane in $\mathbb{P}^n_{\mathbb{F}_q}$ passes through at least one point in P?

Solution 3.9:

Problem 3.10: What is the minimum size of a set of points $P \subseteq \mathbb{P}^3_{\mathbb{F}_2}$, such that every line in $\mathbb{P}^3_{\mathbb{F}_2}$ passes through at least one point in P?

What about for points and lines in $\mathbb{P}^3_{\mathbb{F}_q}$?

What about for points and lines in $\mathbb{P}^n_{\mathbb{F}_q}$?

Solution 3.10:

UCLA Mathematics Department, Los Angeles, CA 90095-1555, USA.
Email address: fzamora@math.princeton.edu

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA.
Email address: jmoraga@math.ucla.edu