Games Packet

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1 Game 1

Play the following game with fellow students. While playing the game, attempt to develop a strategy and determine if either Player 1 or Player 2 has a winning strategy.

There is a pile of n chips on the table. Player 1 can remove any number of chips from 1 to $n-1$. (Player 1 cannot remove all the chips on the first move.) After that, players alternate moving, with each player not allowed to remove more chips than their opponent did on the previous move. The player who removes the last chip wins.

Problem. After playing the game numerous times with fellow students, can you determine if either player has a winning strategy and in what scenarios? Once you have an answer, confirm it works by implementing the strategy in numerous games with a fellow student. Then finally prove to an instructor your solution by presenting it and playing the game against them.

2 Fibonacci Numbers

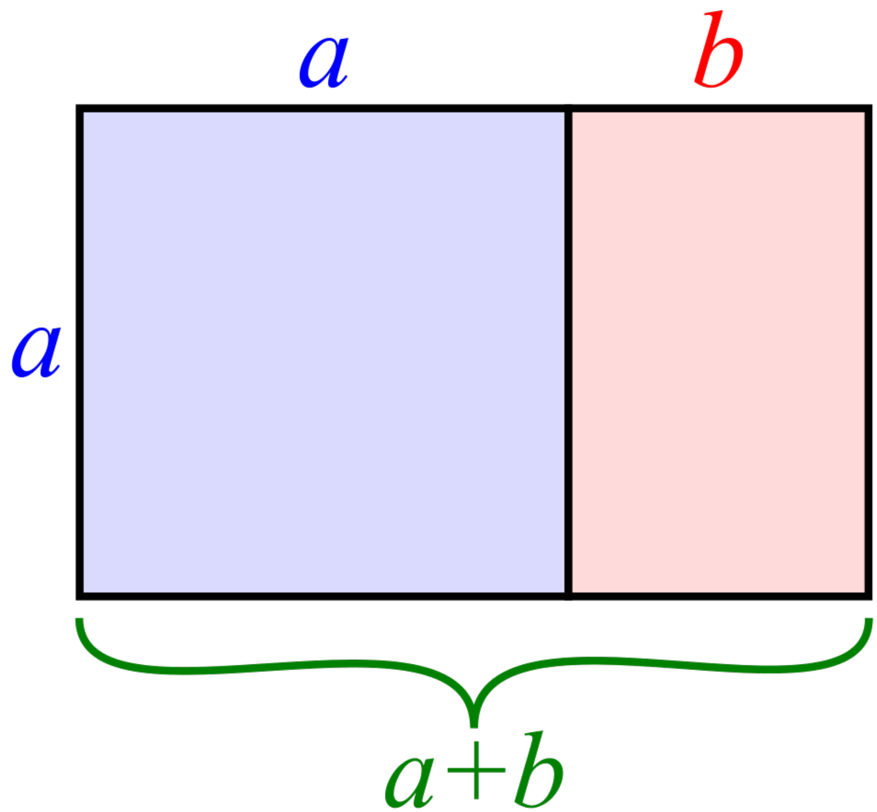
The Fibonacci numbers are a sequence of numbers defined below by

$$F_0 = 0, F_1 = 1, \text{ and } F_{n+1} = F_{n-1} + F_n \text{ for each } n \geq 2$$

and so we can compute a few more terms as so:

$$F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21$$

A closely related number to the Fibonacci numbers, the golden ratio, has been of interest to humanity since Ancient Greece. The golden ratio makes appearences in various patterns in nature and art. These include the spiral arrangement of leaves, other parts of vegetation, and some of Salvador Dali's works. In fact, artists and architects have proportioned their works to approximate the golden ratio due to its aesthetically pleasing form. Today, the golden ratio has even been used to analyze the proportions arising in financial markets.



The golden ratio, often denoted by φ, is the ratio of $a, b > 0$, $\frac{a}{b}$, such that $\frac{a}{b} = \frac{a+b}{a}$. Geometrically, this condition on the ratio between a and b means that the large, exterior rectangle is similar to the smaller right rectangle in the image above. Now by setting $r = \frac{a}{b}$, we can algebraically manipulate the above condition to obtain

$$\frac{a}{b} = \frac{a+b}{a}$$

$$r = 1 + \frac{1}{r}$$

$$r^2 = r + 1$$

$$0 = r^2 - r - 1$$

We can solve for r using the quadratic formula, which gives us that

$$r = \frac{1 \pm \sqrt{5}}{2}$$

The positive version, $\frac{1+\sqrt{5}}{2} \approx 1.618$ is known as the golden ratio and the negative version, $\frac{1-\sqrt{5}}{2} \approx -0.618$ is sometimes referred to as the golden ratio's little brother. We recognize that since $\frac{a}{b}$ is the ratio of positive numbers, it must be positive and therefore the little brother is an extraneous solution while the golden ratio is the true ratio we desire.

A surprising fact about the Fibonacci numbers is that there is an explicit formula for them using the Golden ratio and its little brother. The formula is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for each $n \geq 0$.

Problem. Prove the above formula using mathematical induction.

A surprising result to note here is that as n gets large, $\left(\frac{1-\sqrt{5}}{2}\right)^n$ gets very small since we are raising a base that has absolute value less than 1 to a large power. Thus, as n gets large, $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ gets really close to an integer as it becomes approximately equal to F_n, which is always an integer.

Problem. Show that

$$r = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$$

where $r = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

Problem. Use mathematical induction to show that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

for each $n \geq 1$.

Now we turn to an interesting theorem which uses the Fibonacci numbers. The following is known as Zeckendorf's Theorem.

Theorem 1. Every positive integer can be uniquely represented as a sum of Fibonacci numbers so that the sum does not include any two consecutive Fibonacci numbers.

Example 1.

$$54 = 34 + 13 + 5 + 2$$

Example 2.

$$48 = 34 + 13 + 1$$

Problem. Use mathematical induction to prove the existence part of Zeckendorf's Theorem. That is, show that every positive integer can be represented as a sum of Fibonacci numbers so that the sum does not include any two consecutive Fibonacci numbers.

Problem. Find such a decomposition of the number 121.

Problem. Find such a decomposition of the number 232.

Problem. Find such a decomposition of the number 987.

3 Game 2

Just as before, play the following game with fellow students. While playing, develop some strategies and attempt to determine if either player has a winning strategy. The Zeckendorf Theorem shown earlier in the packet may be helpful in developing a strategy for this game. There is a pile of n chips on the table. Player 1 can remove any number of chips from 1 to $n-1$. (Player 1 cannot remove all the chips on the first move.) After that, players alternate moving, with each player not allowed to remove more than **twice** as many chips as their opponent did on the previous move. The player who removes the last chip wins.

Problem. After playing the game numerous times with fellow students, can you determine if either player has a winning strategy and in what scenarios? Once you have an answer, confirm it works by implementing the strategy in multiple games with a fellow student.