Expected Values

ORMC

01/21/24

1 Introduction

The most important tool in all of probability theory is this:

Theorem 1.1 (Linearity of Expectation)**.** *The* expectation *of a random variable X is a fancy word for its average. If X takes possible values $\{a_1, \dots, a_n\}$, and takes value a_i with probability p_i, then the expectation is $\mathbb{E}[X] = \sum_i p_i a_i$.*

Expectation is linear*:*

- *If X, Y are random variables, then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$*
- *If X is a random variable and c a constant, then $\mathbb{E}[cX] = c\mathbb{E}[X]$.*

This worksheet is motivated by this Putnam problem. If you didn't get to it last week, you can try it now, or try some easier problems and get back to it.

Problem 1.2 (Putnam 2023 B3)**.** A sequence $y_1, y_2, \dots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \dots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \dots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \dots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \dots, i_k$ such that $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \dots, X_n)$ for $n \geq 2$.

Problem 1.3 (HMMT 2006)**.** At a nursery, 2006 babies sit in a circle. Suddenly, each baby randomly pokes either the baby to its left or to its right. What is the expected value of the number of unpoked babies?

2 An AIME Problem

Here's a counting problem where I recommend thinking probabilistically, and using linearity of expectation. First, here's a useful identity of binomial coefficients:

Problem 2.1 (Vandermonde's Identity)**.** Prove that

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

Problem 2.2 (2022 AIME 1 Problem 12)**.** For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \cdots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. What is $\frac{S_{2022}}{S_{2021}}$?

3 The Probabilistic Method

The probabilistic method uses probability calculations to show existence of things. Basically, if the probability of something happening is positive, that thing happens. If the expected value of a random variable is more than x, then the variable is sometimes more than x.

Problem 3.1 (BAMO 2004 Problem 4)**.** Suppose one is given n real numbers, not all zero, but such that their sum is zero. Prove that one can label these numbers $a_1, a_2, \ldots, a_n$ in such a manner that

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 < 0.$$

Problem 3.2. Suppose there are n blue vertices and n red vertices in a graph. The graph is bipartite: no blue vertices are connected to each other, and no red vertices are connected to each other. The graph has at least $n^2 - n + 1$ edges. Show that there is a perfect matching: a set of n edges that hits each vertex exactly once.

4 IMO (and Shortlist)

Problem 4.1 (IMO 1987 Problem 1)**.** Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ which have exactly k fixed points. Prove that $\sum_{k=0}^{n} kp_n(k) = n!$.

Problem 4.2 (IMO 1987 Shortlist)**.** Prove that there exists a 4-coloring of the set $M = \{1, 2, 3, \cdots, 1987\}$ such that any 10-term arithmetic progression in the set M is not monochromatic.

5 Putnam

Problem 5.1 (Putnam 2014 A4)**.** Suppose X is a random variable that takes on only nonnegative integers values, with $\mathbb{E}[X] = 1$, $\mathbb{E}[X2] = 2$, and $\mathbb{E}[X3] = 5$. (Here $\mathbb{E}[Y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.