ORMC AMC 10/12 Group
Winter, Week 3: Modular Arithmetic

Jan 21, 2024

1 Warm-up Exercises

1. **(2014 AMC 8 #21)** The 7-digit numbers $\underline{74A52B1}$ and $\underline{326AB4C}$ are each multiples of 3. What is the sum of all possible values of C?

2. **(2008 AMC 8 #11)** The number 64 has the property that it is divisible by its unit digit. How many whole numbers between 10 and 50 have this property?

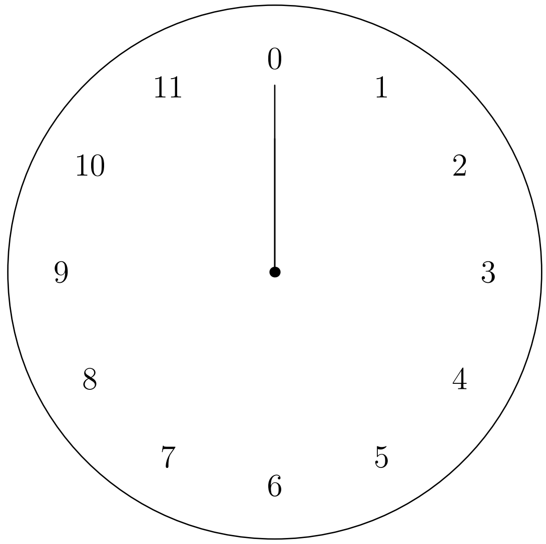
3. **(2016 AMC 8 #24)** The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

4. **(2018 AMC 8 #7)** The 5-digit number $\underline{2}\ \underline{0}\ \underline{1}\ \underline{8}\ \underline{U}$ is divisible by 9. What is the remainder when this number is divided by 8?

5. **(MathCounts)** What is the largest multiple of 12 that can be written using each digit $0, 1, 2, \ldots, 9$ exactly once?

2 Modular Arithmetic

As opposed to doing arithmetic on an infinite number line, like we usually do, think of **modular arithmetic** as doing arithmetic on a "ring" of numbers, like a clock.



The important property is that the clock loops back on itself at a certain point, so as the hour hand ticks around, it'll pass $1, 2, 3, \ldots, 10, 11, 12$, but instead of continuing on to $13, 14, \ldots$, it'll go back to 1.

In the case of the clock, we would call 12 the **modulus**, since it's the point where the numbers loop back around. In particular, we're treating each number the same as its remainder when divided by the modulus, 12. For example, if someone says "it's been 32 hours since 3 o'clock," we can figure out what time it says on the clock by doing $3 + 32 = 35$, and then finding the remainder of 35 divided by 12, which is 11.

Notice that in that example, we have a sort of equivalence between 11 and 35. Just by looking at the clock, we wouldn't be able to tell whether the time being counted is 11 or 35. We say that these numbers are **congruent modulo 12**, and we write it like $\boxed{11 \equiv 35 \pmod{12}}$.

Notice also that instead of a 12, the clock above has a 0. This is because the remainder when we divide 12 by 12 is 0, and more importantly, 12 behaves just like 0 in this arithmetic. For example, $3+12 \equiv 15 \equiv 3 \equiv 3+0 \pmod{12}$ – so we use 0, to make it clear that 12 behaves like our usual 0, within this ring.

In general, the modulus doesn't have to be 12 and it can be any positive integer m that we want. But the important properties we mentioned above remain the same:

$$a \equiv b \pmod{m} \quad \text{if and only if} \quad (a-b) \text{ is divisible by } m \tag{1}$$

$$m \equiv 0 \pmod{m}, \quad \text{and } a + m \equiv a + 0 \equiv a \pmod{m} \tag{2}$$

We also find that addition, subtraction, and multiplication work in pretty much the same way that we are used to for regular arithmetic:

$$\text{if } a \equiv b \pmod{m} \quad \text{and} \quad c \equiv d \pmod{m}:$$

$$a + c \equiv b + d \pmod{m} \tag{3}$$

$$a - c \equiv b - d \pmod{m} \tag{4}$$

$$a \cdot c \equiv b \cdot d \pmod{m} \tag{5}$$

[Bonus: Show that equations (3), (4), and (5) can be derived from (1)]

3 Exercises

1. Derive the divisibility tests for 3 and 9 by using modular arithmetic

   [Hint: the number 46368 can also be written as $4 \cdot 10^4 + 6 \cdot 10^3 + 3 \cdot 10^2 + 6 \cdot 10 + 8$.]

2. The divisibility test for 7 says that a number $N = x \cdot 10 + d$, where d is a single digit and x is an integer, is divisible by 7 when $x - 2d$ is divisible by 7. Prove this using modular arithmetic.

   [i.e., if $N = 46368$, then x would be 4636, and d would be 8]

3. Create a divisibility test for 13, in the same style as the divisibility test for 7.

   [i.e., $N = x \cdot 10 + d$ is divisible by 13 when $x - kd$ is divisible by 13. Find k.]

4. **(2021 Fall AMC 12A #10)** The base-nine representation of the number N is $27{,}006{,}000{,}052_{\text{nine}}$. What is the remainder when N is divided by 5?

5. **(2010 AMC 10B #16)** Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \ldots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3?

6. **(2009 AMC 10B #21)** What is the remainder when $3^0 + 3^1 + 3^2 + \cdots + 3^{2009}$ is divided by 8?

7. **(2010 AIME I #2)** Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.

8. **(1999 AMC 8 #24)** What is the remainder when 1999^{2000} is divided by 5?

9. **(2008 AMC 12A #15)** Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

10. **(Fermat's Little Theorem, Part 1)** Recall the binomial theorem:

$$(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k}$$

Show that if p is a prime, then $(x+y)^p \equiv x^p + y^p \pmod{p}$.

11. **(Fermat's Little Theorem, Part 2)** Using problem 10 and the fact that $0^p \equiv 0 \pmod{p}$, conclude that $a^p \equiv a \pmod{p}$ for all integers a. In particular, if a is not divisible by p, then $\boxed{a^{p-1} \equiv 1 \pmod{p}}$.

12. **(2017 AMC 10B #14)** An integer N is selected at random in the range $1 \leq N \leq 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1?

13. **(2018 AMC 10B #16)** Let $a_1, a_2, \dots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

14. **(Wilson's Theorem, Part 1)** Notice that Fermat's Little Theorem tells us that, modulo p, every number $k \in 1, 2, \dots, p-1$ has an inverse k^{-1}, such that $k \cdot k^{-1} \equiv 1 \pmod{p}$. Show that if $k^{-1} = k$, then $k = 1$ or $p - 1$.

15. **(Wilson's Theorem, Part 2)** Using the result of problem 14, show that $\boxed{(p-1)! \equiv -1 \pmod{p}}$.

16. **(2002 ARML)** Let a be an integer such that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{23} = \frac{a}{23!}.$$

Find the remainder when a is divided by 13.

17. If p is a prime greater than 2, define $p = 2q + 1$. Prove that $(q!)^2 + (-1)^q$ is divisible by p.

18. **(1995 AHSME #27)** Consider the triangular array of numbers with 0,1,2,3,... along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 2 & & & \\
 & & 3 & & 4 & & 4 & & 3 & & \\
 & 4 & & 7 & & 8 & & 7 & & 4 & \\
5 & & 11 & & 15 & & 15 & & 11 & & 5
\end{array}
$$

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100?

19. **(2006 AMC 10B #25)** Mr. Jones has eight children of different ages. On a family trip his oldest child, who is 9, spots a license plate with a 4-digit number in which each of two digits appears two times. "Look, daddy!" she exclaims. "That number is evenly divisible by the age of each of us kids!" "That's right," replies Mr. Jones, "and the last two digits just happen to be my age." Which of the following is not the age of one of Mr. Jones's children?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

20. **(2019 AMC 10A #25)** For how many integers n between 1 and 50, inclusive, is $\frac{(n^2-1)!}{(n!)^n}$ an integer?

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35