Intro to Proofs

Prepared by Mark on April 3, 2024

Part 1:

Problem 1:
We say an integer x is *even* if $x = 2k$ for some $k \in \mathbb{Z}$. We say x is *odd* if $x = 2k + 1$ for some $k \in \mathbb{Z}$. Assume that every integer is even or odd, and never both.

- Show that the product of two odd integers is odd.
- Let $a, b \in \mathbb{Z}, a \neq 0$. We say a *divides* b and write $a \mid b$ if there is a $k \in \mathbb{Z}$ so that $ak = b$.
  Show that $a \mid b \implies a \mid 2b$
- Show that $5n^2 + 3n + 7$ is odd for any $n \in \mathbb{Z}$.
- Let a, b, c be integers so that $a^2 + b^2 = c^2$.
  Show that one of a, b is even.
- Show that every odd integer is the difference of two squares.
- Prove the assumption in the statement of this problem.

Problem 2:
Let $r \in \mathbb{R}$. We say r is *rational* if there exist $p, q \in \mathbb{Z}, q \neq 0$ so that $r = \frac{a}{b}$

- Show that $\sqrt{2}$ is irrational.

- Show that the product of two rational numbers must be rational, while the product of irrational numbers may be rational or irrational. If you claim a number is irrational, provide a proof.

Problem 3:
Let $X = \{x \in \mathbb{Z} \mid x \geq 2\}$. For $k \geq 2$, define $X_k = \{kx \mid x \in X\}$.
What is $X - (X_2 \cup X_3 \cup X_4 \cup ...)$? Prove your claim.

Problem 4:
Show that there are infinitely may primes.
You may use the fact that every integer has a prime factorization.

Problem 5:
For a set X, define its *diagonal* as $\mathrm{D}(X) = \{(x,x) \in X \times X \mid x \in X\}$.

An *undirected graph* G is an ordered pair (V, E), where V is a set, and $E \subset V \times V$ satisfies $(a,b) \in E \iff (b,a) \in E$ and $E \cap \mathrm{D}(X) = \varnothing$.
The elements of V are called *vertices*; the elements of E are called *edges*.

- Make sense of the conditions on E.

- The *degree* of a vertex a is the number of edges connected to that vertex.
  We'll denote this as $d(a)$. Write a formal definition of this function using set-builder notation and the definitions above. Recall that $|X|$ denotes the size of a set X.

- There are 9 people at a party. Show that they cannot each have 3 friends.
  Friendship is always mutual.

Problem 6:
Let f be a function from a set X to a set Y. We say f is *injective* if $f(x) = f(y) \implies x = y$.
We say f is *surjective* if for all $y \in Y$ there exists an $x \in X$ so that $f(x) = y$.
Let A, B, C be sets, and let $f : A \to B$, $g : B \to C$ be functions. Let $h = g \circ f$.

- Show that if h is injective, f must be injective and g may not be injective.
- Show that if h is surjective, g must be surjective and f may not be surjective.

Problem 7:
Let $X = \{1, 2, ..., n\}$ for some $n \geq 2$. Let $k \in \mathbb{Z}$ so that $1 \leq k \leq n - 1$.
Let $E = \{Y \subset X \mid |Y| = k\}$, $E_1 = \{Y \in E \mid 1 \in Y\}$, and $E_2 = \{Y \in E \mid 1 \notin Y\}$

- Show that $\{E_1, E_2\}$ is a partition of E.
  In other words, show that $\varnothing \neq E_1$, $\varnothing \neq E_2$, $E_1 \cup E_2 = E$, and $E_1 \cap E_2 = \varnothing$.
  Hint: What does this mean in English?

- Compute $|E_1|$, $|E_2|$, and $|E|$.
  Recall that a set of size n has $\binom{n}{k}$ subsets of size k.

- Conclude that for any n and k satisfying the conditions above,

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

- For $t \in \mathbb{N}$, show that $\binom{2t}{t}$ is even.

Theorem 8: The Division Algorithm
Given two integers a, b, we can find two integers q, r, where $0 \leq r < b$ and $a = qb + r$.
In other words, we can divide a by b to get q remainder r.

Problem 9:
Let $x, y \in \mathbb{N}$ be natural numbers. Consider the set $S = \{ax + by \mid a, b \in \mathbb{Z}, ax + by > 0\}$.
The well-ordering principle states that every nonempty subset of the natural numbers has a least element.

- Show that S has a least element. Call it d.
- Let $z = \gcd(x, y)$. Show that z divides d.
- Show that d divides x and d divides y.
- Prove or disprove $\gcd(x, y) \in S$.

Problem 10:

- Let $f : X \to Y$ be an injective function. Show that for any two functions $g : Z \to X$ and $h : Z \to X$, if $f \circ g = f \circ h$ from Z to Y then $g = h$ from Z to X.
  By definition, functions are equal if they agree on every input in their domain.
  Hint: This is a one-line proof.

- Let $f : X \to Y$ be a surjective function. Show that for any two functions $g : Y \to W$ and $h : Y \to W$, if $g \circ f = h \circ f \implies g = h$.

⋆ Let $f : X \to Y$ be a function where for any set Z and functions $g : Z \to X$ and $h : Z \to X$, $f \circ g = f \circ h \implies g = h$. Show that f is injective.

⋆ Let $f : X \to Y$ be a function where for any set W and functions $g : Y \to W$ and $h : Y \to W$, $g \circ f = h \circ f \implies g = h$. Show f is surjective.

Problem 11:
In this problem we prove the binomial theorem: for $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{N-k}$$

In the proof below, we let a and b be arbitrary numbers.

- Check that this formula works for $n = 0$. Also, check a few small n to get a sense of what's going on.
- Let $N \in \mathbb{N}$. Suppose we know that for a specific value of N,

  $$(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}$$

  Now, show that this formula also works for $N + 1$.
- Conclude that this formula works for all $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

Problem 12:
A *relation* on a set X is an $R \subset X \times X$.

- We say R is *reflexive* if $(x, x) \in R$ for all $x \in X$.
- We say R is *symmetric* if $(x, y) \in R \implies (y, x) \in R$.
- We say R is *transitive* if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$.
- We say R is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Say we have a set X and an equivalence relation R.
The *equivalence class* of an element $x \in X$ is the set $\{y \in X \mid (x, y) \in R\}$.

Let R be an equivalence relation on a set X.
Show that the set of equivalence classes is a partition of X.

Problem 13:
Show that there exist two positive irrational numbers a and b so that a^b is rational.

Problem 14:
Show that the following holds for any planar graph:

$$\text{vertices} - \text{edges} + \text{faces} = 2$$

Hint: If you don't know what these words mean, ask an instructor.

Problem 15:
Consider a rectangular chocolate bar of arbitrary size.
What is the minimum number of breaks you need to make to seperate all its pieces?

Problem 16:
Four travellers are on a plane, each moving along a straight line at an arbitrary constant speed.
No two of their paths are parallel, and no three intersect at the same point.
We know that traveller A has met traveler B, C, and D, and that B has met C and D (and A).
Show that C and D must also have met.

Problem 17:
Say we have an n-gon with non-intersecting edges.
What is the size of the smallet set of vertices that can "see" every point inside the polygon?