OLGA RADKO MATH CIRCLE, WINTER 2024: ADVANCED 3

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Worksheet 2:

Throughout this worksheet $\mathbb{F}$ is a field.

Remember that the *Projective Space* $\mathbb{P}^n_{\mathbb{F}}$ is defined to be the set of points with coordinates $[x_0 : x_1 : \ldots : x_n]$, with $x_i \in \mathbb{F}$ not all of them equal to 0, where two sets of coordinates $[x_0 : \ldots : x_n]$ and $[y_0 : \ldots y_n]$ define the same point if there exists a constant $\lambda \in \mathbb{F}$, such that for all i:

$$x_i = \lambda y_i.$$

Problem 2.1: Count how many points there are in $\mathbb{P}^n_{\mathbb{F}_q}$ in two different ways:

- Decompose into disjoint projective spaces of lower dimension, by separating in cases where a coordinate x_i is zero or non-zero.
- Notice that the possible coordinates of projective plane are $\mathbb{F}^{n+1} \setminus (0, \ldots, 0)$, and count how many different points of $\mathbb{F}^{n+1}$ represent the same point of projective space.

Solution 2.1:

A line in projective plane $\mathbb{P}^n_{\mathbb{F}}$ can be defined by a linear projective parametrization with parameters r, s, i.e. the points in a line are those that have coordinates

$$[p_0(r,s) : \ldots : p_n(r,s)]$$

where p_i are homogeneous degree 1 polynomials over $\mathbb{F}$, and r, s take values in $\mathbb{F}$ other than $r = s = 0$, such that the polynomials do not all vanish at the same time.

Problem 2.2: Determine the points of the following lines:

(1) $[r+s : r : s]$ in $\mathbb{P}^2_{\mathbb{F}_3}$
(2) $[r+s : r : s : r-s]$ in $\mathbb{P}^3_{\mathbb{F}_3}$
(3) $[r+s : r : s : r-s : 0]$ in $\mathbb{P}^4_{\mathbb{F}_3}$
(4) $[r+s : r : s]$ in $\mathbb{P}^2_{\mathbb{F}_5}$

Can you see how many points does a line in $\mathbb{P}^n_{\mathbb{F}_q}$ have?

Solution 2.2:

A hyperplane in projective space $\mathbb{P}^n_{\mathbb{F}}$ is the zero-set of a degree one polynomial, i.e. it is the elements $[x_0 : \ldots : x_n]$ satisfying the equation:

$$a_0x_0 + \ldots + a_nx_n = 0,$$

For some fixed $a_i \in \mathbb{F}$.

Problem 2.3: Find all the points and give a parametrization for the lines that are given by the intersections of the following planes:

(1) $x_0 = 0$ and $x_1 = 0$ in $\mathbb{P}^3_{\mathbb{F}_2}$
(2) $x_0 + x_1 = 0$ and $x_2 = 0$ in $\mathbb{P}^3_{\mathbb{F}_5}$
(3) $x_0 - 2x_1 = 0$ and $x_2 - x_1 + 3x_3 = 0$ in $\mathbb{P}^3_{\mathbb{F}_7}$

Solution 2.3:

In general a d-dimensional projective space in $\mathbb{P}^n_{\mathbb{F}}$ can be defined as a linear projective parametrization, with parameters $r_0, \ldots, r_d$, i.e. the points are those that have coordinates:

$$[p_0(r_0, \ldots, r_d) : \ldots : p_n(r_0, \ldots, r_d))]$$

Problem 2.4: Determine the dimension and find a parametrization for the following projective spaces:

(1) $x_1 = 0$ in $\mathbb{P}^2_{\mathbb{F}_2}$
(2) $x_0 = 0$ in $\mathbb{P}^4_{\mathbb{F}_2}$
(3) The intersection of $x_1 + x_3 = 0$ and $x_2 - x_0 = 0$ in $\mathbb{P}^3_{\mathbb{F}_3}$
(4) The intersection of $x_0 = 0$, $x_1 = 0$, $x_2 = 0$ and $x_0 - x_1 + x_2 = 0$ in $\mathbb{P}^5_{\mathbb{F}_7}$

Solution 2.4:

Problem 2.5: What is the dimension of a hyperplane in $\mathbb{P}^n_{\mathbb{F}}$?

What is the dimension of the intersection of two different hyperplanes in $\mathbb{P}^n_{\mathbb{F}}$?

What are the possible dimensions of the intersection a line and a hyperplane in $\mathbb{P}^3_{\mathbb{F}}$?

In general what are the possible dimensions of the intersection of a d-dimensional and an r-dimensional projective space inside of $\mathbb{P}^n_{\mathbb{F}}$?

Solution 2.5:

Problem 2.6: How many different lines are there in $\mathbb{P}^3_{\mathbb{F}_q}$?

How many different hyperplanes are there in $\mathbb{P}^3_{\mathbb{F}_q}$?

How many different hyperplanes are there in $\mathbb{P}^n_{\mathbb{F}_q}$?

Solution 2.6:

$\mathbb{F}^n$ is also called the *affine space* $\mathbb{A}^n_{\mathbb{F}}$. This can be regarded as a subset of $\mathbb{P}^n_{\mathbb{F}}$, by setting the first coordinate $x_0 = 1$.

Then lines, planes, hyperplanes and lower-dimensional affine spaces in $\mathbb{A}^n_{\mathbb{F}}$ can be regarded as the restrictions of lines, planes, hyperplanes and lower dimensional projective spaces in $\mathbb{P}^n_{\mathbb{F}}$

Problem 2.7: Do two lines in $\mathbb{A}^2_{\mathbb{F}}$ always intersect?

At how many points can two planes in $\mathbb{A}^3_{\mathbb{F}_q}$ intersect?

Solution 2.7:

Problem 2.8: In general in how many points can two hyperplanes intersect in $\mathbb{A}^n_{\mathbb{F}_q}$
Solution 2.8:

Problem 2.9: How many different planes through a point are there in $\mathbb{A}^4_{\mathbb{F}_q}$?

Can you notice any relation between this quantity and the number of lines in $\mathbb{P}^3_{\mathbb{F}_q}$?

Solution 2.9:

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