Worksheet 2:

Throughout this worksheet $\mathbb{F}$ is a field.

Remember that the Projective Space $\mathbb{P}^n_\mathbb{F}$ is defined to be the set of points with coordinates $[x_0 : x_1 : \ldots : x_n]$, with $x_i \in \mathbb{F}$ not all of them equal to 0, where two sets of coordinates $[x_0 : \ldots : x_n]$ and $[y_0 : \ldots y_n]$ define the same point if there exists a constant $\lambda \in \mathbb{F}$, such that for all $i$:

$$x_i = \lambda y_i.$$  

**Problem 2.1:** Count how many points there are in $\mathbb{P}^n_\mathbb{F}$ in two different ways:

- Decompose into disjoint projective spaces of lower dimension, by separating in cases where a coordinate $x_i$ is zero or non-zero.

- Notice that the possible coordinates of projective plane are $\mathbb{F}^{n+1} \setminus (0, \ldots, 0)$, and count how many different points of $\mathbb{F}^{n+1}$ represent the same point of projective space.

**Solution 2.1:**
A line in projective plane $\mathbb{P}^n_F$ can be defined by a linear projective parametrization with parameters $r, s$, i.e. the points in a line are those that have coordinates 

$$[p_0(r, s) : \ldots : p_n(r, s)]$$

where $p_i$ are homogeneous degree 1 polynomials over $\mathbb{F}$, and $r, s$ take values in $\mathbb{F}$ other than $r = s = 0$, such that the polynomials do not all vanish at the same time.

**Problem 2.2:** Determine the points of the following lines:

1. $[r + s : r : s]$ in $\mathbb{P}^2_{\mathbb{F}_3}$
2. $[r + s : r : s : r - s]$ in $\mathbb{P}^3_{\mathbb{F}_3}$
3. $[r + s : r : s : r - s : 0]$ in $\mathbb{P}^4_{\mathbb{F}_3}$
4. $[r + s : r : s]$ in $\mathbb{P}^2_{\mathbb{F}_5}$

Can you see how many points does a line in $\mathbb{P}^n_{\mathbb{F}_q}$ have?

**Solution 2.2:**
A hyperplane in projective space \( \mathbb{P}^n_F \) is the zero-set of a degree one polynomial, i.e. it is the elements \([x_0 : \ldots : x_n]\) satisfying the equation:

\[
a_0x_0 + \ldots + a_nx_n = 0,
\]

For some fixed \(a_i \in F\).

**Problem 2.3:** Find all the points and give a parametrization for the lines that are given by the intersections of the following planes:

(1) \(x_0 = 0\) and \(x_1 = 0\) in \(\mathbb{P}^3_{\mathbb{F}_2}\)

(2) \(x_0 + x_1 = 0\) and \(x_2 = 0\) in \(\mathbb{P}^3_{\mathbb{F}_5}\)

(3) \(x_0 - 2x_1 = 0\) and \(x_2 - x_1 + 3x_3 = 0\) in \(\mathbb{P}^3_{\mathbb{F}_7}\)

**Solution 2.3:**
In general a $d$-dimensional projective space in $\mathbb{P}_F^n$ can be defined as a linear projective parametrization, with parameters $r_0, \ldots, r_d$, i.e. the points are those that have coordinates:

$$[p_0(r_0, \ldots, r_d) : \ldots : p_n(r_0, \ldots, r_d)]$$

**Problem 2.4:** Determine the dimension and find a parametrization for the following projective spaces:

1. $x_1 = 0$ in $\mathbb{P}_{\mathbb{F}_2}^2$
2. $x_0 = 0$ in $\mathbb{P}_{\mathbb{F}_2}^4$
3. The intersection of $x_1 + x_3 = 0$ and $x_2 - x_0 = 0$ in $\mathbb{P}_{\mathbb{F}_3}^3$
4. The intersection of $x_0 = 0$, $x_1 = 0$, $x_2 = 0$ and $x_0 - x_1 + x_2 = 0$ in $\mathbb{P}_{\mathbb{F}_7}^5$

**Solution 2.4:**
Problem 2.5: What is the dimension of a hyperplane in $\mathbb{P}^n_k$?
What is the dimension of the intersection of two different hyperplanes in $\mathbb{P}^n_k$?
What are the possible dimensions of the intersection a line and a hyperplane in $\mathbb{P}^n_k$?
In general what are the possible dimensions of the intersection of a $d$-dimensional and an $r$-dimensional projective space inside of $\mathbb{P}^n_k$?
Solution 2.5:
Problem 2.6: How many different lines are there in $\mathbb{P}^3_{\mathbb{F}_q}$?

How many different hyperplanes are there in $\mathbb{P}^3_{\mathbb{F}_q}$?

How many different hyperplanes are there in $\mathbb{P}^n_{\mathbb{F}_q}$?

Solution 2.6:
\( \mathbb{F}^n \) is also called the *affine space* \( \mathbb{A}_F^n \). This can be regarded as a subset of \( \mathbb{P}_F^n \), by setting the first coordinate \( x_0 = 1 \).

Then lines, planes, hyperplanes and lower-dimensional affine spaces in \( \mathbb{A}_F^n \) can be regarded as the restrictions of lines, planes, hyperplanes and lower dimensional projective spaces in \( \mathbb{P}_F^n \).

**Problem 2.7:** Do two lines in \( \mathbb{A}_F^2 \) always intersect?

At how many points can two planes in \( \mathbb{A}_F^3 \) intersect?

**Solution 2.7:**
Problem 2.8: In general in how many points can two hyperplanes intersect in $\mathbb{A}^p_{\mathbb{F}_q}$
Solution 2.8:
Problem 2.9: How many different planes through a point are there in $A^3_{\mathbb{F}_q}$?

Can you notice any relation between this quantity and the number of lines in $P^3_{\mathbb{F}_q}$?

Solution 2.9:
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