Geometry and Group Theory Part II: The Fundamental Group

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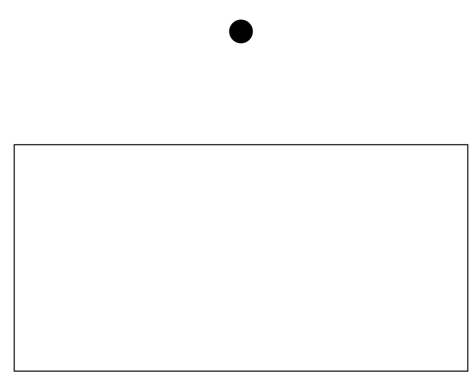
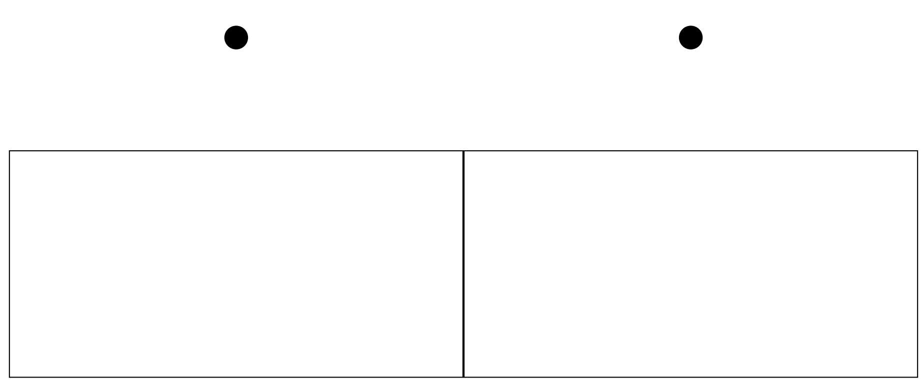
Advanced 1 - Winter 2024

1 Warm-Up: Hanging Paintings

Problem 1 *Consider the following painting (the rectangle in the diagram below), which lies on a wall with a nail (the black dot). If the floor is below this diagram, the painting would just fall down, so we can attach a string to the painting to prevent this. The string will go from the top left of the painting to the top right. Draw an example of such a string that would keep the painting from falling to the ground.*



Problem 2 *Draw two potential ways for the string to* ***not*** *stop the paintings from falling. We consider two such strings to be the* ***same*** *if we can move the painting and string around on the wall (but not through the nail) to get the other configuration. Are the two strings you drew the same?*

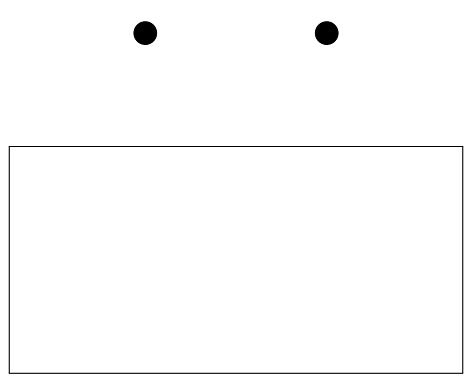


Problem 3 *Explain why any two ways to not have the painting hanging are the same. Can a hanging painting be the same as a not hanging painting?*

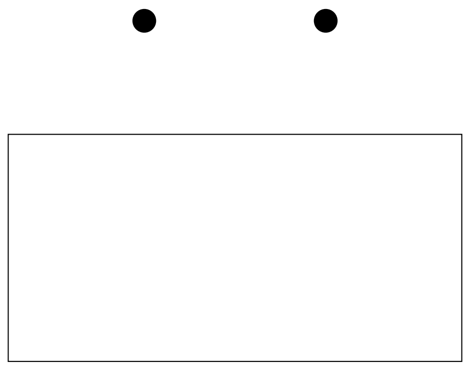
We say a painting *depends* on a nail (or subset of nails) if it is hanging, but removing that nail (or subset of nails) will make it fall to the floor.

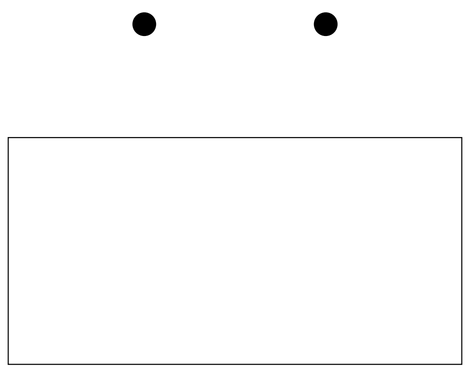
Problem 4 *Explain why a painting hanging on one nail always depends on that nail.*

Problem 5 *Consider the following painting on a wall with two nails. Can you draw a string that depends on neither nail?*

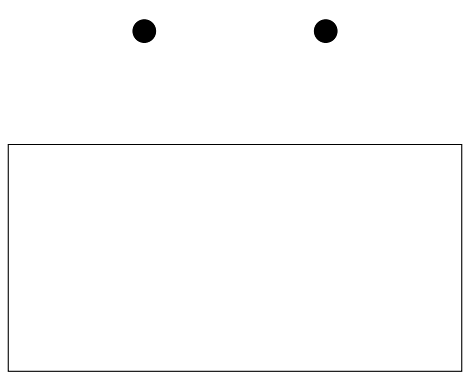


Problem 6 *Can you draw a string that depends on the left nail, but not the right? How about the right but not the left?*





Problem 7 *Can you draw a string that depends on both nails individually? (That is, removing either nail will drop the painting.)*



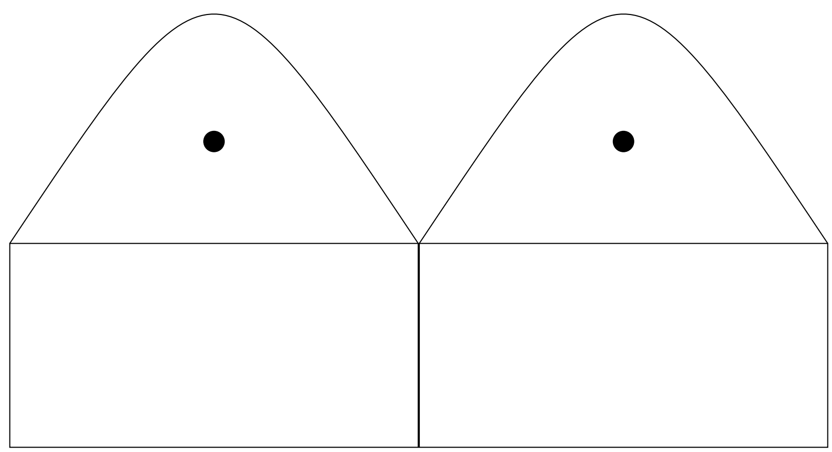
2 A Reminder about Groups

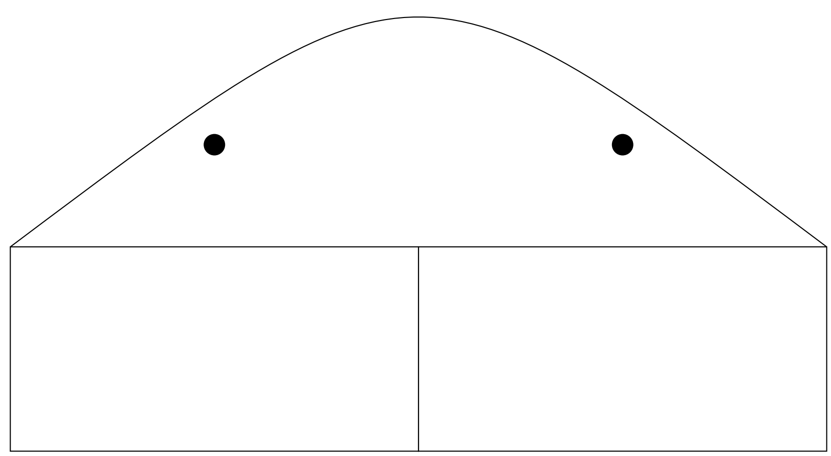
Last week, we defined a *group*, whose definition we'll repeat below.

Definition 1 *A **group** is a set G together with a multiplication operation $\cdot$ such that*

- *(Associativity) For all $x, y, z \in G$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.*
- *(Identity) There is an element $e \in G$ such that for any $x \in G$, $e \cdot x = x \cdot e = x$. e is called the **identity element** of G.*
- *(Inverses) For all $x \in G$, there exists a $y \in G$ such that $x \cdot y = y \cdot x = e$.*

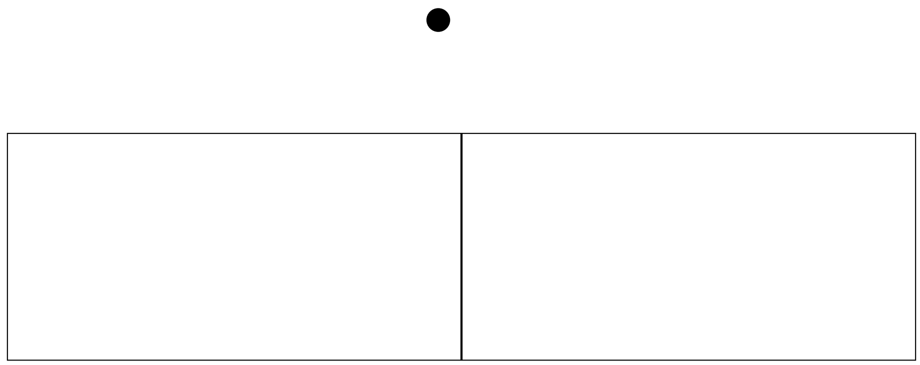
As you may be able to guess, the different ways to hang paintings will form a group. First, we need to define what it means to multiply two paintings. Since our string goes from the top left of the painting to the top right, we can place painting A next to painting B, and painting AB will be the glued painting where the string follows A's string from start to end, then B's string. In the following example of multiplying two paintings hanging on two nails, the product is shown in the second picture below..



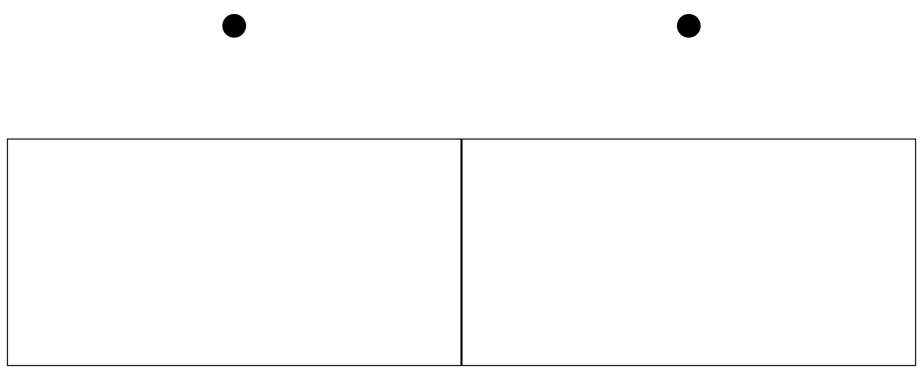


Note that we have moved around the string in the product a little bit. As discussed earlier, that keeps the painting the same.

Problem 8 *Draw the product of the two paintings you drew in Problem* 2*. Does the result hang or fall?*



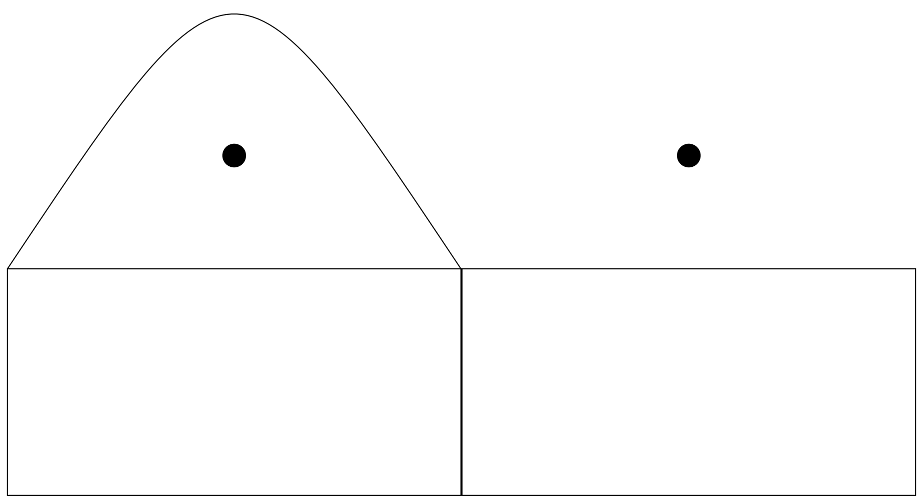
Problem 9 *Draw the product of the paintings you drew in Problem* 6*. Does the result still depend on either nail (or both)?*



Problem 10 *Explain why multiplication of paintings is associative (that is, why it satisfies the associativity condition of Definition* 1*).*

Problem 11 *For any possible number of nails, is there an identity painting (that is, a painting satisfying the identity condition of Definition* 1*)? If so, draw it below and explain why it is the identity.*

Problem 12 *Let's consider the one nail case for now. Given the following painting on the left, draw its inverse painting on the right.*



Problem 13 *Given a general painting on one nail, what is its inverse painting?*

3 Multiple Nails and Words

Problem 14 *(Bonus) Recall our definition of* ***isomorphism*** *from last week, which is a way of saying whether two groups are the same. Show that the group of paintings with one nail is isomorphic to the group $\mathbb{Z}$ of integers with the operation $+$.*

We now move on to studying the case with more than one nail.

Definition 2

- *A* ***word*** *is a string of letters and their inverses from a given finite set (referred to as an* ***alphabet****). For example, in the alphabet $\{a, b\}$, aba an $abba^{-1}b^{-1}$ are words.*
- *A* ***reduced*** *(or simplified) word is a word where we combine adjacent letters that are the same by adding their exponents. For example, baa^{-1} reduces to b and $abba^{-1}b^{-1}$ reduces to $ab^2a^{-1}b^{-1}$.*
- *Words can be multiplied by concatenating them (putting them together) as strings. The* ***free group*** *on an alphabet is the set of all possible reduced words in that alphabet with this multiplication operation.*

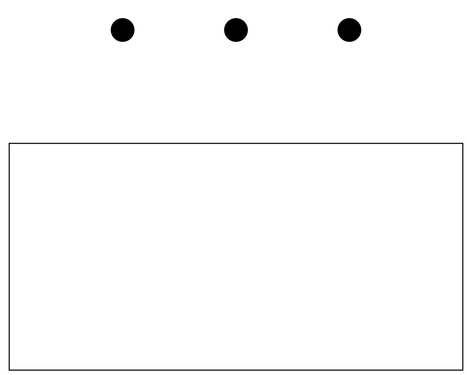
Problem 15 *The definition of multiplication in a free group seems a lot like our definition of multiplying paintings. Can you think of a way to correspond paintings (on, say, two nails) with words in some alphabet?*

Problem 16 *Show that the free group on any alphabet is a group. Also show that the free group on an alphabet with one letter is isomorphic to the integers with the operation $+$.*

Problem 17 *(Bonus) Show that the group of paintings on n nails is isomorphic to the free group on an alphabet with n letters.*

Problem 18 *Translate our definition of a painting depending on a nail into the setting of a free group. What does such a painting correspond to as a word?*

Problem 19 *Using Problem* 18*, come up with a way to hang a painting on three nails such that it depends on every nail. If you would like, also draw it below.*



Problem 20 *Can you think of a way to hang a painting on any finite number n nails, such that it depends on every nail?*