Selected 2023 Putnam Problems

ORMC

01/14/24

Here are half of the problems from the Putnam exam that happened this December. Putnam problems are tricky, so I've given you some help. There are some problems that serve as hints, and some theorems that you may find useful. If you haven't seen these theorems before, you can either use them without proof, or try to prove them.

Of course, I also have more hints on everything - just ask.

1 Binary Expansions

Problem 1.1. Which positive integers can be written with only a single one in their binary representation?

Problem 1.2. For each positive integer n, let $k(n)$ be the number of ones in the binary representation of $7 \cdot n$. What is the minimum value of $k(n)$?

Theorem 1.3 (Chinese Remainder Theorem)**.** *Let m, n be relatively prime positive integers, and let a, b be integers. Then there is some integer c with $c \equiv a \pmod{m}$ and $c \equiv b \pmod{n}$.*

Theorem 1.4 (Euler's Theorem)**.** *Let $\phi(n)$ be the number of integers between 1 and n (inclusive) that are relatively prime to n. Then for any integer a, $a^{\phi}(n) \equiv 1 \pmod{n}$.*

Problem 1.5. What is $\phi(17^2)$?

Problem 1.6. Factorize 2023.

Problem 1.7 (Putnam 2023 B2)**.** For each positive integer n, let $k(n)$ be the number of ones in the binary representation of $2023 \cdot n$. What is the minimum value of $k(n)$?

2 Chessboard Combinatorics

Problem 2.1 (Putnam 2023 B1)**.** Consider an m-by-n grid of unit squares, indexed by (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i, j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i, j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1, j), (i, j+1)$, and $(i+1, j+1)$ are unoccupied, then a legal move is to slide the coin from (i, j) to $(i+1, j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

Hint: In a valid configuration, what do the *unoccupied* squares look like?

3 Polynomials

We've already seen lots of useful tricks with polynomials - this problem can be solved with some of those.

Problem 3.1 (Putnam 2023 A2)**.** Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \dots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$.

4 Probability

The most important tool in all of probability theory is this:

Theorem 4.1 (Linearity of Expectation)**.** *The* expectation *of a random variable X is a fancy word for its average. If X takes possible values $\{a_1, \dots, a_n\}$, and takes value a_i with probability p_i, then the expectation is $\mathbb{E}[X] = \sum_i p_i a_i$.*

Expectation is linear*:*

- *If X, Y are random variables, then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$*
- *If X is a random variable and c a constant, then $\mathbb{E}[cX] = c\mathbb{E}[X]$.*

Problem 4.2 (Putnam 2023 B3)**.** A sequence $y_1, y_2, \dots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \dots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \dots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \dots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \dots, i_k$ such that $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \dots, X_n)$ for $n \geq 2$.

5 Trig Derivatives

This one requires some calculus knowledge - if you know how to take derivatives, this is a good one to try.

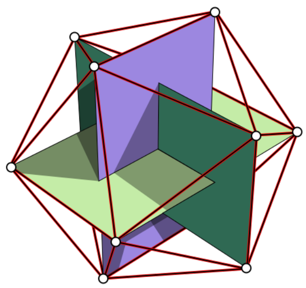
Problem 5.1. For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. What is $f_n'(0)$?

Problem 5.2 (Putnam 2023 A1)**.** For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.

6 Vectors and Rational Approximations

An icosahedron is a convex 3D shape made of 20 equilateral triangles. The logo of the Putnam exam is an icosahedron, so there are a lot of icosahedron problems.

Problem 6.1. In this picture of an icosahedron, what is the ratio between the sides of the rectangles?



Problem 6.2. Prove that if $a \in \mathbb{R}$ is irrational, then for any $x \in \mathbb{R}$ and $\varepsilon > 0$, there are integers m, n such that $|ma - n| < \varepsilon$.

This problem requires some background on vector addition - if you're not familiar with it, I can explain it.

Problem 6.3. Let $v_1, \ldots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \ldots, a_{12}$ such that $\|a_1 v_1 + \cdots + a_{12} v_{12} - v\| < \varepsilon$.