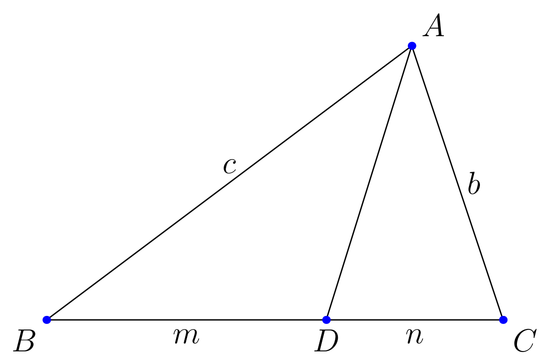
ORMC AMC 10/12 Group
Winter, Week 2: Mass Points

Jan 14, 2024

1 Warm-up Exercises

1. In triangle ABC, let AD be the angle bisector of $\angle BAC$, where D is on BC.



   Show that the ratio of areas $[ABD] : [ACD]$ is the same as the ratio $AB : AC$.

2. **(Angle Bisector Theorem)** Show that when AD bisects $\angle BAC$, $\frac{m}{n} = \frac{c}{b}$.
   (Hint: Use the result of Problem 1.)

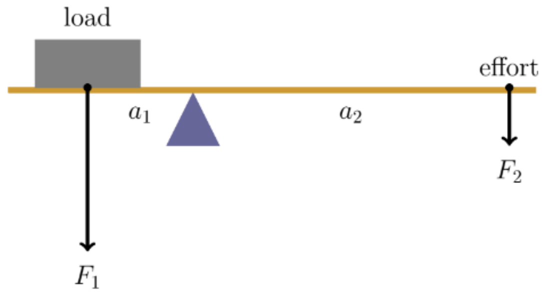
3. Show that when AD bisects $\angle BAC$, $AD^2 = bc - mn$.

4. Show that if ABC is isosceles, then

$$a\cos(B) + b\cos(C) + c\cos(A) = \frac{a+b+c}{2}.$$

2 Mass Points

Consider a lever, as depicted below. It is a rigid beam which pivots around 1 point– its *fulcrum* When you apply force (down) on one side of the lever, this results in force being exerted in the opposite direction (up), on the other side of the lever. The distance from the fulcrum to some force F is called the *arm* of the force.

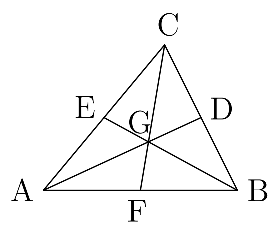


In the diagram above, the arm of force F_1 is a_1, and the arm of F_2 is a_2. The product $T = aF$ is called the "moment of force", or "***torque***." The lever is balanced when the two torques are equal:

$$\boxed{a_1T_1 = a_2T_2} \qquad \Longleftrightarrow \qquad \boxed{\frac{T_2}{T_1} = \frac{a_1}{a_2}}.$$

The idea of mass points is that we can apply this lever rule to shapes – usually triangles. Suppose we have a triangle with some cevians and some given lengths. We can start by choosing a center of mass for the triangle which will act like a fulcrum. Then, we assign weights to the vertices so that the torques are balanced across the center of mass. These weights then allow us to deduce other lengths of the triangle, according to the torque law above.

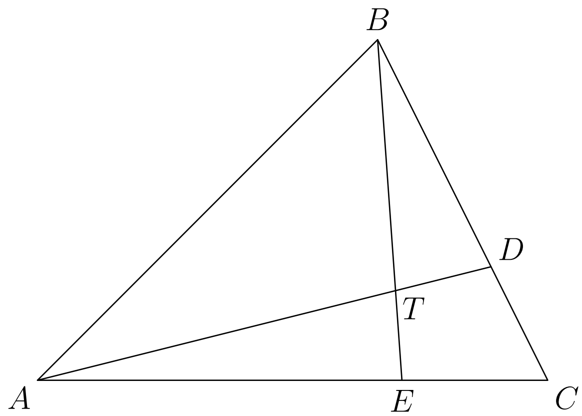
Example: Consider $\triangle ABC$ with medians AD, BE, and CF. Let the centroid be G. What are the ratios $AG : GD, BG : GE, CG : GF$?
Solution:



1. Choose G to be the center of mass of the triangle. The triangle should now balance *across* any line passing through G, meaning that the torques on either side of such a line are equal. And it should also balance *along* any such line, meaning that torques on the line, with G as the fulcrum, are equal.
2. Assign a weight to some vertex– for example, let's say the weight at vertex A is 1.
3. Balancing the triangle *across* line CF, we see that C has an arm of 0, and the arms of A and B are equal, since CF is a median. So, to balance the torques, vertex B has the same weight as A, which is 1. Doing the same for line BE tells us that the weight at vertex C is also 1.
4. Notice that F is the center of mass of A and B, since they balance there. So, even though there is no actual weight at F, we can treat F as having an effective weight of 2, and use it to account for both A and B, as opposed to dealing with them separately. We can do the same for E and D.
5. Balancing the triangle *along* line CF, we see that C has a weight of 1 on one end, and F has a weight of 2 on the other end. Since F has double the weight of C, the torques can only be balanced if C has double the arm of F. So, $\boxed{CG : GF = 2 : 1}$. The previous steps tell us that we can do the same along AD and BE, so $\boxed{AG : GD = BG : GE = 2 : 1}$.

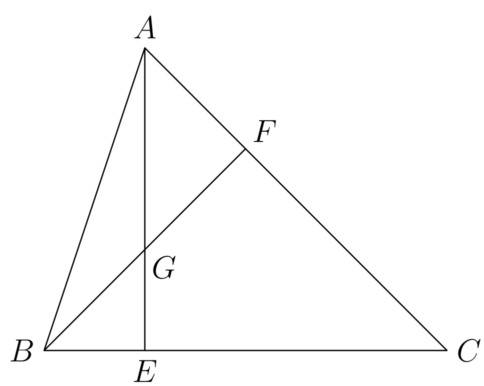
3 Exercises

1. **(2004 AMC 10B #20)** In $\triangle ABC$ points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?



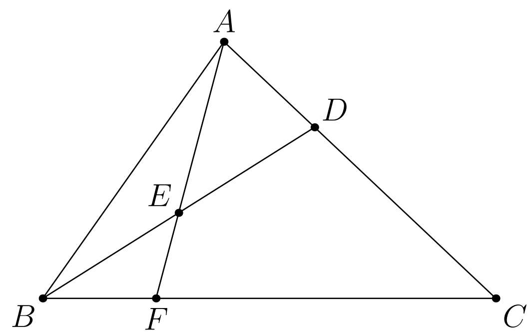
2. **(2011 AIME II #4)** In triangle ABC, $AB = 20$ and $AC = 11$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. The ratio of CP to PA can be expressed in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. **(1971 AHSME #26)** In $\triangle ABC$, point F divides side AC in the ratio $1 : 2$. Let E be the point of intersection of side BC and AG where G is the midpoints of BF. Find $BE : EC$.



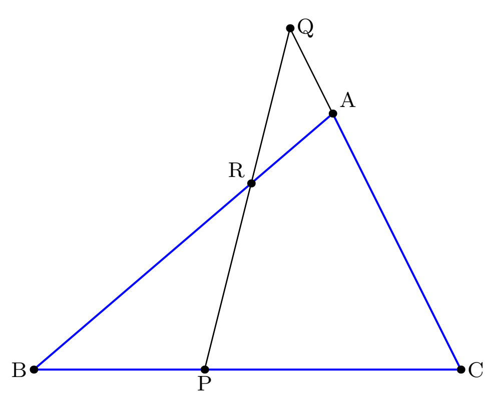
4. **(1965 AHSME #37)** Point E is selected on side AB of $\triangle ABC$ in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC such that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Find $\frac{EF}{FC} + \frac{AF}{FD}$.

5. **(2019 AMC 8 #24)** In triangle $\triangle ABC$, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line $\overline{BC}$ and line $\overline{AE}$. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



6. **(2016 AMC 12A #12)** In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?

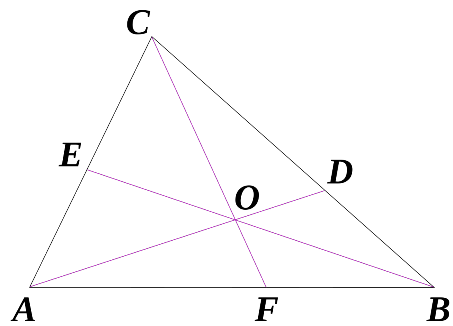
7. **(Menelaus's Theorem)** In the diagram below, $\frac{PB}{CP} \cdot \frac{x}{y} \cdot \frac{AR}{RB} = 1$, where x, y are segments in the diagram. Find x and y.



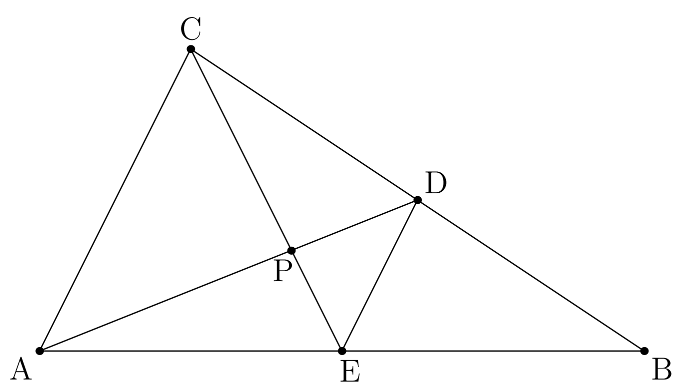
8. **(Ceva's Theorem)** Show that in triangle ABC with cevians AD, BE, CF, if the cevians all intersect at one point G, then

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

(Bonus: Show the converse)

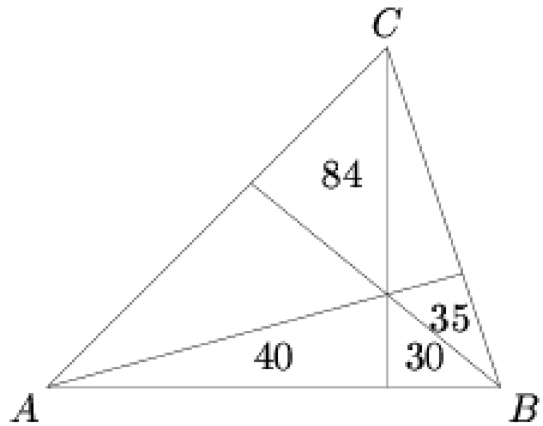


9. **(2013 AMC 10B #16)** In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?

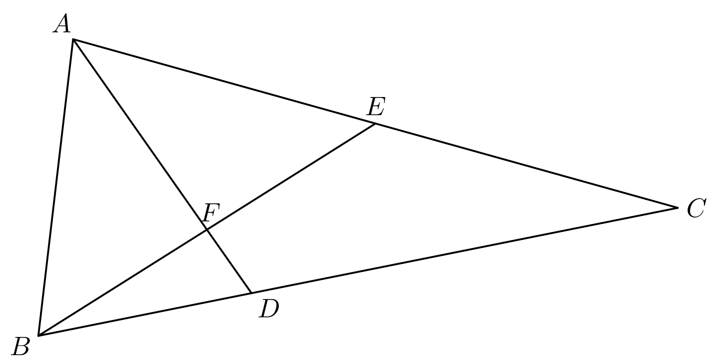


10. **(2001 AIME I #7)** Triangle ABC has $AB = 21$, $AC = 22$ and $BC = 20$. Points D and E are located on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$ and contains the center of the inscribed circle of triangle ABC. Then $DE = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

11. **(1985 AIME #6)** As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



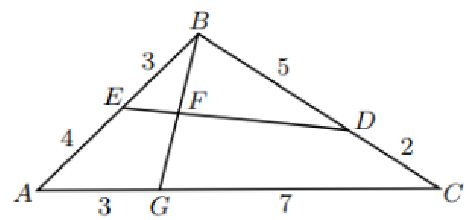
12. **(1980 AHSME #21)** In triangle ABC, $\measuredangle CBA = 72^\circ$, E is the midpoint of side AC, and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. Find the ratio of the area of triangle BDF to the area of quadrilateral $FDCE$.



13. **(2016 AMC 10A #19)** In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of r, s, and t is 1. What is $r + s + t$?

14. **(2009 AIME I #5)** Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

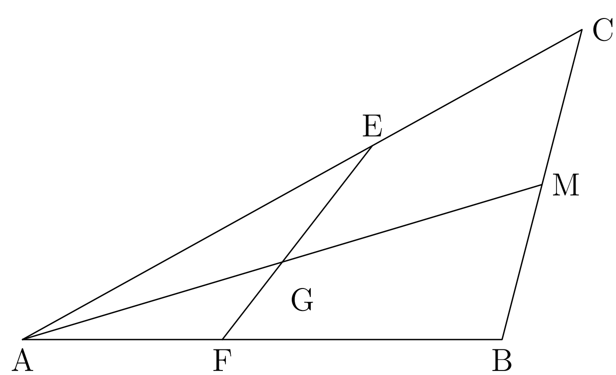
15. In the figure below, ED joins points E and D on the sides of $\triangle ABC$ forming a transversal. Cevian BG divides AC in a ratio of 3 to 7 and intersects the transversal ED at point F. Find the ratios $EF : FD$ and $BF : FG$.



16. **(Varignon's Theorem)** Show that the midpoints of the edges of an arbitrary quadrilateral are the vertices of a parallelogram.

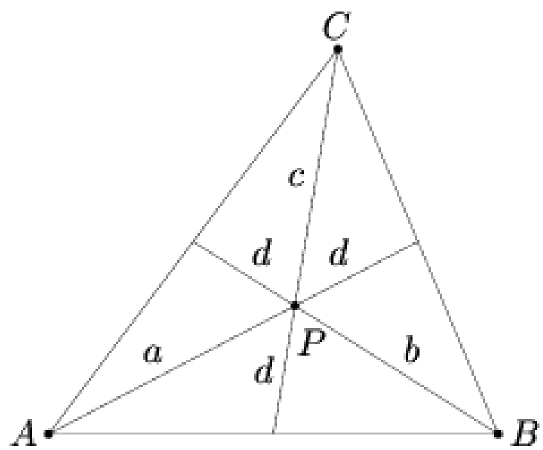
(Hint: assign a mass of 1 to each vertex, and find the quadrilateral's center of mass in 2 different ways.)

17. **(1975 AHSME #28)** In $\triangle ABC$ shown in the adjoining figure, M is the midpoint of side BC, $AB = 12$ and $AC = 16$. Points E and F are taken on AC and AB, respectively, and lines EF and AM intersect at G. If $AE = 2AF$, then find $\frac{EG}{GF}$.



18. **(1992 AIME #14)** In triangle ABC, A', B', and C' are on the sides BC, AC, and AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

19. **(1988 AIME #12)** Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$.



20. **(1989 AIME #15)** Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB (see the figure below). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.