1. In triangle $ABC$, let $AD$ be the angle bisector of $\angle BAC$, where $D$ is on $BC$.

Show that the ratio of areas $[ABD] : [ACD]$ is the same as the ratio $AB : AC$.

2. (Angle Bisector Theorem) Show that when $AD$ bisects $\angle BAC$, $\frac{m}{n} = \frac{c}{b}$.

(Hint: Use the result of Problem 1.)

3. Show that when $AD$ bisects $\angle BAC$, $AD^2 = bc - mn$.

4. Show that if $ABC$ is isosceles, then

$$a \cos(B) + b \cos(C) + c \cos(A) = \frac{a + b + c}{2}.$$
2 Mass Points

Consider a lever, as depicted below. It is a rigid beam which pivots around 1 point– its fulcrum. When you apply force (down) on one side of the lever, this results in force being exerted in the opposite direction (up), on the other side of the lever. The distance from the fulcrum to some force $F$ is called the arm of the force.

In the diagram above, the arm of force $F_1$ is $a_1$, and the arm of $F_2$ is $a_2$. The product $T = aF$ is called the “moment of force”, or “torque.” The lever is balanced when the two torques are equal:

$$a_1T_1 = a_2T_2 \iff \frac{T_2}{T_1} = \frac{a_1}{a_2}$$

The idea of mass points is that we can apply this lever rule to shapes – usually triangles. Suppose we have a triangle with some cevians and some given lengths. We can start by choosing a center of mass for the triangle which will act like a fulcrum. Then, we assign weights to the vertices so that the torques are balanced across the center of mass. These weights then allow us to deduce other lengths of the triangle, according to the torque law above.

**Example:** Consider $\triangle ABC$ with medians $AD, BE,$ and $CF$. Let the centroid be $G$. What are the ratios $AG:GD,BG:GE,CG:GF$?

**Solution:**

1. Choose $G$ to be the center of mass of the triangle. The triangle should now balance across any line passing through $G$, meaning that the torques on either side of such a line are equal. And it should also balance along any such line, meaning that torques on the line, with $G$ as the fulcrum, are equal.

2. Assign a weight to some vertex– for example, let’s say the weight at vertex $A$ is 1.

3. Balancing the triangle across line $CF$, we see that $C$ has an arm of 0, and the arms of $A$ and $B$ are equal, since $CF$ is a median. So, to balance the torques, vertex $B$ has the same weight as $A$, which is 1. Doing the same for line $BE$ tells us that the weight at vertex $C$ is also 1.

4. Notice that $F$ is the center of mass of $A$ and $B$, since they balance there. So, even though there is no actual weight at $F$, we can treat $F$ as having an effective weight of 2, and use it to account for both $A$ and $B$, as opposed to dealing with them separately. We can do the same for $E$ and $D$.

5. Balancing the triangle along line $CF$, we see that $C$ has a weight of 1 on one end, and $F$ has a weight of 2 on the other end. Since $F$ has double the weight of $C$, the torques can only be balanced if $C$ has double the arm of $F$. So, $[CG:GF = 2:1]$. The previous steps tell us that we can do the same along $AD$ and $BE$, so $[AG:GD = BG:GE = 2:1]$.
3 Exercises

1. **(2004 AMC 10B #20)** In $\triangle ABC$ points $D$ and $E$ lie on $BC$ and $AC$, respectively. If $AD$ and $BE$ intersect at $T$ so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?

![Diagram 1](image1.png)

2. **(2011 AIME II #4)** In triangle $ABC$, $AB = 20$ and $AC = 11$. The angle bisector of $\angle A$ intersects $BC$ at point $D$, and point $M$ is the midpoint of $AD$. Let $P$ be the point of the intersection of $AC$ and $BM$. The ratio of $CP$ to $PA$ can be expressed in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

![Diagram 2](image2.png)

3. **(1971 AHSME #26)** In $\triangle ABC$, point $F$ divides side $AC$ in the ratio $1 : 2$. Let $E$ be the point of intersection of side $BC$ and $AG$ where $G$ is the midpoints of $BF$. Find $BE : EC$.

![Diagram 3](image3.png)
4. (1965 AHSME #37) Point $E$ is selected on side $AB$ of $\triangle ABC$ in such a way that $AE : EB = 1 : 3$ and point $D$ is selected on side $BC$ such that $CD : DB = 1 : 2$. The point of intersection of $AD$ and $CE$ is $F$. Find $\frac{EF}{FC} + \frac{AF}{FD}$.

5. (2019 AMC 8 #24) In triangle $\triangle ABC$, point $D$ divides side $AC$ so that $AD : DC = 1 : 2$. Let $E$ be the midpoint of $BD$ and let $F$ be the point of intersection of line $BC$ and line $AE$. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?

6. (2016 AMC 12A #12) In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point $D$ lies on $BC$, and $AD$ bisects $\angle BAC$. Point $E$ lies on $AC$, and $BE$ bisects $\angle ABC$. The bisectors intersect at $F$. What is the ratio $AF : FD$?

7. (Menelaus’s Theorem) In the diagram below, $\frac{PB}{CP} \cdot \frac{x}{y} \cdot \frac{AR}{RB} = 1$, where $x, y$ are segments in the diagram. Find $x$ and $y$. 

---

A
B
C
P
Q
R

x
y
8. **(Ceva’s Theorem)** Show that in triangle $ABC$ with cevians $AD, BE, CF$, if the cevians all intersect at one point $G$, then
\[
\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.
\]
(Bonus: Show the converse)

9. **(2013 AMC 10B #16)** In triangle $ABC$, medians $AD$ and $CE$ intersect at $P$, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?

10. **(2001 AIME I #7)** Triangle $ABC$ has $AB = 21$, $AC = 22$ and $BC = 20$. Points $D$ and $E$ are located on $AB$ and $AC$, respectively, such that $DE$ is parallel to $BC$ and contains the center of the inscribed circle of triangle $ABC$. Then $DE = m/n$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$. 
11. **(1985 AIME #6)** As shown in the figure, triangle $ABC$ is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle $ABC$.

12. **(1980 AHSME #21)** In triangle $ABC$, $\angle CBA = 72^\circ$, $E$ is the midpoint of side $AC$, and $D$ is a point on side $BC$ such that $2BD = DC$; $AD$ and $BE$ intersect at $F$. Find the ratio of the area of triangle $BDF$ to the area of quadrilateral $FDCE$.

13. **(2016 AMC 10A #19)** In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point $E$ between $B$ and $C$, and point $F$ between $E$ and $C$ are such that $BE = EF = FC$. Segments $AE$ and $AF$ intersect $BD$ at $P$ and $Q$, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of $r$, $s$, and $t$ is 1. What is $r + s + t$?
14. (2009 AIME I \#5) Triangle $ABC$ has $AC = 450$ and $BC = 300$. Points $K$ and $L$ are located on $AC$ and $AB$ respectively so that $AK = CK$, and $CL$ is the angle bisector of angle $C$. Let $P$ be the point of intersection of $BK$ and $CL$, and let $M$ be the point on line $BK$ for which $K$ is the midpoint of $PM$. If $AM = 180$, find $LP$.

15. In the figure below, $ED$ joins points $E$ and $D$ on the sides of $\triangle ABC$ forming a transversal. Cevian $BG$ divides $AC$ in a ratio of 3 to 7 and intersects the transversal $ED$ at point $F$. Find the ratios $EF : FD$ and $BF : FG$.

16. (Varignon’s Theorem) Show that the midpoints of the edges of an arbitrary quadrilateral are the vertices of a parallelogram.

(Hint: assign a mass of 1 to each vertex, and find the quadrilateral’s center of mass in 2 different ways.)

17. (1975 AHSME \#28) In $\triangle ABC$ shown in the adjoining figure, $M$ is the midpoint of side $BC$, $AB = 12$ and $AC = 16$. Points $E$ and $F$ are taken on $AC$ and $AB$, respectively, and lines $EF$ and $AM$ intersect at $G$. If $AE = 2AF$, then find $\frac{EG}{GF}$.
18. **(1992 AIME #14)** In triangle $ABC$, $A'$, $B'$, and $C'$ are on the sides $BC$, $AC$, and $AB$, respectively. Given that $AA'$, $BB'$, and $CC'$ are concurrent at the point $O$, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

19. **(1988 AIME #12)** Let $P$ be an interior point of triangle $ABC$ and extend lines from the vertices through $P$ to the opposite sides. Let $a$, $b$, $c$, and $d$ denote the lengths of the segments indicated in the figure. Find the product $abc$ if $a + b + c = 43$ and $d = 3$.

20. **(1989 AIME #15)** Point $P$ is inside $\triangle ABC$. Line segments $APD$, $BPE$, and $CPF$ are drawn with $D$ on $BC$, $E$ on $AC$, and $F$ on $AB$ (see the figure below). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$. 

![Diagram](image)