Combinatorics Problems

ORMC

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1 Book Problems: Putnam and Beyond

Problem 1.1. If X is a finite set, show that you can list the subsets of X in such a way that

- the empty set comes first
- each subset occurs exactly once
- each set differs from the next set in the list by either adding or deleting a single element.

Problem 1.2. Let M be a subset of $\{1, \dots, 15\}$ such that the product of any three distinct elements of M is not a square. What is the maximum size of M?

Hint: Look for disjoint triples with square products.

1.1 Graph Theory

Problem 1.3. Prove that every graph has two vertices that are endpoints of the same number of edges.

Problem 1.4. In a society of n mathematicians, any two mathematicians who do not know each other have exactly two common coauthors, and any two mathematicians who know each other don't have other common coauthors. Prove that in this society every mathematician has the same number of coauthors.

1.2 Permutations

Problem 1.5. For each permutation $a_1, a_2, \dots, a_{10}$ of the integers $1, 2, 3, \dots, 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|.$$

Find the average value of all such sums.

Problem 1.6. Let $f(n)$ be the number of permutations $a_1, a_2, \dots, a_n$ of the integers $1, 2, \dots, n$ such that

- $a_1 = 1$
- $a_n = n$
- for $i < n$, $|a_i - a_{i+1}| \leq 2$.

Determine whether $f(2024)$ is divisible by 3.

Hint: Try finding a recursion relation for $f(n)$.

Bonus: What if we remove the condition $a_n = n$?

2 Competition Problems

Problem 2.1 (BAMO 2022 Problem C1)**.** The game of pool includes 15 balls in a triangle number arrangement. 7 are striped, and 8 are solid colors. Prove that no matter how the 15 balls are arranged in the rack, there must always be a pair of striped balls adjacent to each other.

Problem 2.2 (BAMO 2015 Problem E1)**.** For $n > 1$, consider an $n \times n$ chessboard and place pieces at the centers of different squares.

- With $2n$ chess pieces on the board, show that there are 4 pieces among them that form the vertices of a parallelogram.
- Show that there is a way to place $2n - 1$ chess pieces so that no 4 of them form the vertices of a parallelogram.

Problem 2.3 (USAMO 2006 Problem 2)**.** For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k + 1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $N/2$.

Problem 2.4 (USAMO 2003 Problem 6)**.** At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Problem 2.5 (Putnam 2003 A1)**.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.