Geometry and Group Theory Part I: Symmetries

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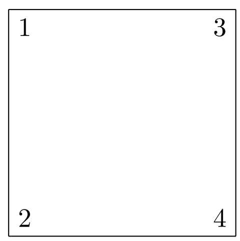
Advanced 1 - Winter 2024

1 Symmetries

Consider the following square, which has its vertices labelled. We'll call a geometric move that takes the square back to the same position (but not necessarily preserving the numbers of the vertices) a *symmetry* of the square.



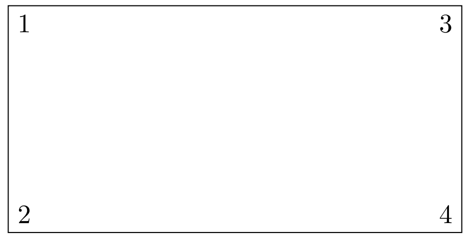
We'll call (for now) the set of symmetries of the square its *symmetry group*.

Problem 1 *Suppose the square starts glued to the table, so that we can rotate it but not reflect it. List all the elements of the symmetry group.*

Given two symmetries S and T, we can *multiply* them to make the symmetry ST, by applying T first, then applying S.

Problem 2 *Write down the* ***multiplication table*** *for the symmetries you found in Problem 1 - that is, write down the result of the multiplication of any pair of symmetries.*

Problem 3 *Suppose the following rectangle starts glued to the table, so that we can only rotate but not reflect it. Give the symmetries and their multiplication table.*



Problem 4 *Now suppose we cut the rectangle out of the page, so that we can reflect it as well. Give all the possible symmetries now, and their multiplication table.*

Now that we have two examples of symmetry groups, let us do the natural thing and compare them. We'll say that two groups are the *isomorphic* (a formal way of saying that they're the "same") if there is a renaming of one group's elements that makes its multiplication table the same as the other group's.

Problem 5 *In Problems 2 and 4, we wrote down the multiplication tables of two four-element groups. Are those groups the same? Why or why not?*

We now take a look at some bigger symmetry groups.

Problem 6 *Suppose we have a regular octagon glued to the table (as in Problems 1 and 3). Describe its symmetry group. (Hint: You don't need to give the full multiplication table, but at least describe how multiplication works.)*

Problem 7 *Suppose the square from Problem* 1 *is no longer glued to the table. Write down its symmetry group with the multiplication table. Is this the same as the octagon?*

2 A Brief Digression: Definitions

So far, we've been calling the sets of symmetries symmetry groups, without actually defining what that meant. We'll do so now, before examining further geometric examples.

Definition 1 *A **group** is a set G together with a multiplication operation $\cdot$ such that*

- *(Associativity) For all $x, y, z \in G$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.*
- *(Identity) There is an element $e \in G$ such that for any $x \in G$, $e \cdot x = x \cdot e = x$. e is called the **identity element** of G.*
- *(Inverses) For all $x \in G$, there exists a $y \in G$ such that $x \cdot y = y \cdot x = e$.*

Problem 8 *The fact that the group operation is called multiplication doesn't mean it has to be multiplication of numbers! As an example, show that the set of integers forms a group with the multiplication operation $+$.*

Problem 9 *Explain why symmetry groups, with the multiplication operation defined previously, have identity and inverse elements. (Bonus) Also explain why this multiplication is associative, so that all symmetry groups are groups.*

From now on, we will omit the $\cdot$ in group multiplication—xy will mean $x \cdot y$. Also, you may have noticed that one familiar property from, say, the integers, is missing from Definition 1.

Definition 2 *We say that $x, y \in G$ are **commutative** (or commute) with each other if $xy = yx$. G is said to be **abelian** if any pair of its elements commute.*

Problem 10 *There is a very good reason we didn't define all groups to be abelian—some of our symmetry groups aren't! Go back and check every group's multiplication table that you wrote down. Which one(s) are nonabelian?*

Finally, we define what it means for two groups to be the same, or isomorphic, which will just be a formal way of stating that there is a renaming of elements that makes the multiplication tables the same.

Definition 3 *Groups G and H are **isomorphic** if there is a one-to-one and onto function $f : G \to H$ (called an isomorphism) such that*

$$f(xy) = f(x)f(y) \text{ for all } x, y \in G$$

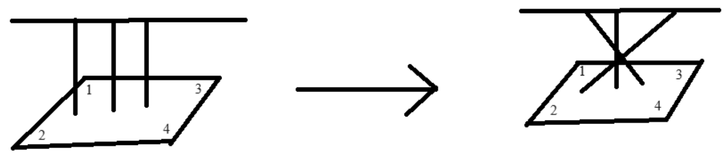
Problem 11 *Show that the integers modulo 4 form a group with the operation $+$. Also, show that this group is isomorphic to the square rotation group you found in Problems 1-2.*

Problem 12 *What familiar group from number theory is the octagon rotation group from Problem 6 isomorphic to? Explain why.*

Problem 13 *Show that if groups G and H are isomorphic and G is abelian, so is H. In particular, this is another way of showing that the groups from Problems 6 and 7 are not isomorphic.*

3 More Examples

Problem 14 *Your instructors will have a physical copy of the following setup. Take a rectangle attached to a stick by three pieces of string. A symmetry will take the rectangle to itself, but not necessarily any of the vertices or the twisting of the string. For example, the following symmetry is* ***not*** *the identity. Write down the symmetry group and multiplication table below.*



Problem 15 *Prove that this is not isomorphic to the integers mod* 8 *(aka the octagon rotation group).*

Problem 16 *Prove that this is not isomorphic to the square symmetry group from Problem* 7*.*

Groups don't have to come from geometric symmetries. One natural symmetry comes from permuting a set of n distinct objects, which fits the loose definition we started with.

Definition 4 *The **symmetric group** on n objects, denoted S_n, is the set of all permutations σ of the n objects, with multiplication $\sigma\tau$ given by doing the permutation τ first, then the permutation σ.*

Problem 17 *Show that S_n is a group. How many elements does it have?*

Problem 18 *Give a way to write every symmetry group in this worksheet as a permutation of some number of elements. This is part of the reason S_n is named the "symmetric group".*

Definition 5 *A **subgroup** H of a group G is a subset of G that forms a group with the same multiplication operator.*

Problem 19 *Use Problem* 18 *to show that each symmetry group in this worksheet is a subgroup of some symmetric group S_n.*

Problem 20 *(Bonus) Show that **any** group with finitely many elements is a subgroup of some symmetric group S_n.*