ORMC AMC 10/12 Group
Winter, Week 1: Trigonometry

Jan 7, 2024

1 Warm-up Exercises

1. Compute

$$\frac{\sin(1)\sin(2)\cdots\sin(89)}{\cos(1)\cos(2)\cdots\cos(89)}$$

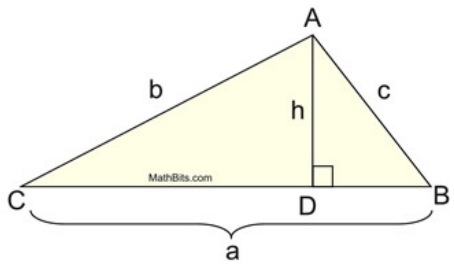
2. If $\sin(x) = \frac{60}{61}$, what is $\cos(x)$? In general, if $0 < x < 90$, what is $\cos(x)$ in terms of $\sin(x)$?

3. **(1988 AHSME #13)** If $\sin(x) = 3\cos(x)$ then what is $\sin(x) \cdot \cos(x)$?
(Hint: recall that $\tan(x) = \sin(x)/\cos(x)$.)

4. **(1999 AHSME #15)** Let C be an acute angle such that $\frac{1}{\cos(C)} - \tan(C) = \frac{1}{2}$. Find $\frac{1}{\cos(C)} + \tan(C)$.

2 Theorems

We know that the area of the triangle below is $\frac{1}{2}a \cdot h$. Alternatively, since the height h makes right triangles, we can express it in terms of trig formulas to get a different area formula:



$$h = b \cdot \sin(C) \implies \boxed{K = \frac{1}{2}a \cdot b\sin(C)}$$

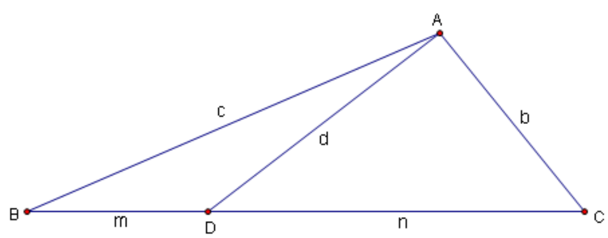
We can get the same area by doing this for the other angles A and B, and then multiplying through by $\frac{2}{abc}$ gives us the extended ***Law of Sines***:

$$K = \frac{1}{2}ab\sin(C) = \frac{1}{2}bc\sin(A) = \frac{1}{2}ac\sin(B) \implies \boxed{\frac{2K}{abc} = \frac{\sin(C)}{c} = \frac{\sin(A)}{a} = \frac{\sin(B)}{b}}.$$

And since h makes two right triangles, we can use the pythagorean theorem to get two equivalent expressions of h^2. If we equate these expressions, noting that $CD = b\cos(C)$, we get the ***Law of Cosines***:

$$h^2 = b^2 - (b\cos(C))^2, \qquad h^2 = c^2 - (a - b\cos(C))^2$$
$$\implies b^2 - b^2\cos^2(C) = c^2 - a^2 + 2ab\cos(C) - b^2\cos^2(C) \implies \boxed{a^2 + b^2 = c^2 - 2ab\cos(C)}$$

Consider the following diagram, where we define $a = m + n$:



The cevian AD gives us two triangles ABC and ADC which share angle C. So, the law of cosines gives us two equivalent expressions for $\cos(C)$. Equating these, we get ***Stewart's Theorem***:

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}, \qquad \cos(C) = \frac{n^2 + b^2 - c^2}{2nb},$$
$$\implies \frac{a^2 + b^2 - c^2}{2ab} = \frac{n^2 + b^2 - d^2}{2nb} \implies na^2 + nb^2 - nc^2 = an^2 + ab^2 - ad^2$$
$$\implies na(a - n) + ad^2 = (a - n)b^2 + nc^2 \implies \boxed{man + ad^2 = mb^2 + nc^2}$$

3 Exercises

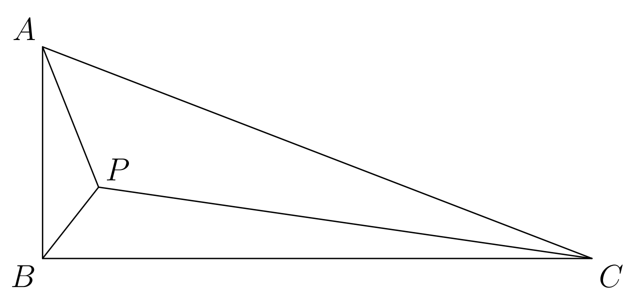
1. In triangle ABC, $AB = 13$, $BC = 14$, and $AC = 15$. Point P is on BC so that AP is perpendicular to BC. What is the length of BP?

2. In triangle ABC, $AB = 13$, $A = 75$, and $B = 45$. What are the perimeter and area of ABC?

3. **(1963 AHSME #34)** In $\triangle ABC$, side $a = \sqrt{3}$, side $b = \sqrt{3}$, and side $c > 3$. Let x be the largest number such that the magnitude, in degrees, of the angle opposite side c exceeds x. Which of the following is x?

   (A) $150°$ **(B)** $120°$ **(C)** $105°$ **(D)** $90°$ **(E)** $60°$

4. **(1987 AIME #9)** Triangle ABC has right angle at B, and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.

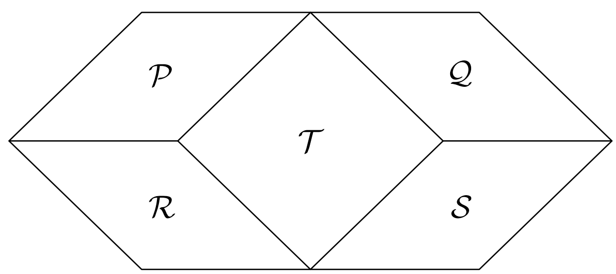


5. **(2003 AMC 12B #21)** An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

6. **(2002 AMC 12B #23)** In $\triangle ABC$, we have $AB = 1$ and $AC = 2$. Side $\overline{BC}$ and the median from A to $\overline{BC}$ have the same length. What is BC?

7. **(2001 AIME I #4)** In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

8. **(2006 AIME I #8)** Hexagon $ABCDEF$ is divided into five rhombuses, $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$, and $\mathcal{T}$, as shown. Rhombuses $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, and $\mathcal{S}$ are congruent, and each has area $\sqrt{2006}$. Let K be the area of rhombus $\mathcal{T}$. Given that K is a positive integer, find the number of possible values for K.



9. **(2012 AMC 12A #16)** Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

10. **(2003 AIME I #7)** Point B is on $\overline{AC}$ with $AB = 9$ and $BC = 21$. Point D is not on $\overline{AC}$ so that $AD = CD$, and AD and BD are integers. Let s be the sum of all possible perimeters of $\triangle ACD$. Find s.

11. **(2017 AMC 12B #15)** Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

12. **(2019 AIME I #3)** In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

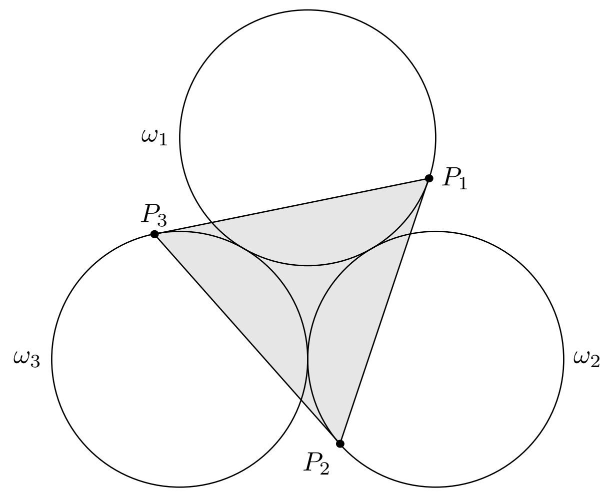
13. **(2019 AMC 12A #19)** In $\triangle ABC$ with integer side lengths, $\cos A = \frac{11}{16}$, $\cos B = \frac{7}{8}$, and $\cos C = -\frac{1}{4}$. What is the least possible perimeter for $\triangle ABC$?

14. **(2022 AMC 10B #20)** Let $ABCD$ be a rhombus with $\angle ADC = 46^\circ$. Let E be the midpoint of $\overline{CD}$, and let F be the point on $\overline{BE}$ such that $\overline{AF}$ is perpendicular to $\overline{BE}$. What is the degree measure of $\angle BFC$?

15. **(2013 AMC 10A #23)** In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

16. **(2013 AIME II #13)** In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

17. **(2018 AMC 12B #25)** Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. See the figure below. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$ for positive integers a and b. What is $a + b$?



18. **(2020 AMC 12A #24)** Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

19. **(2006 AIME I #14)** A tripod has three legs each of length 5 feet. When the tripod is set up, the angle between any pair of legs is equal to the angle between any other pair, and the top of the tripod is 4 feet from the ground. In setting up the tripod, the lower 1 foot of one leg breaks off. Let h be the height in feet of the top of the tripod from the ground when the broken tripod is set up. Then h can be written in the form $\frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $\lfloor m + \sqrt{n} \rfloor$.