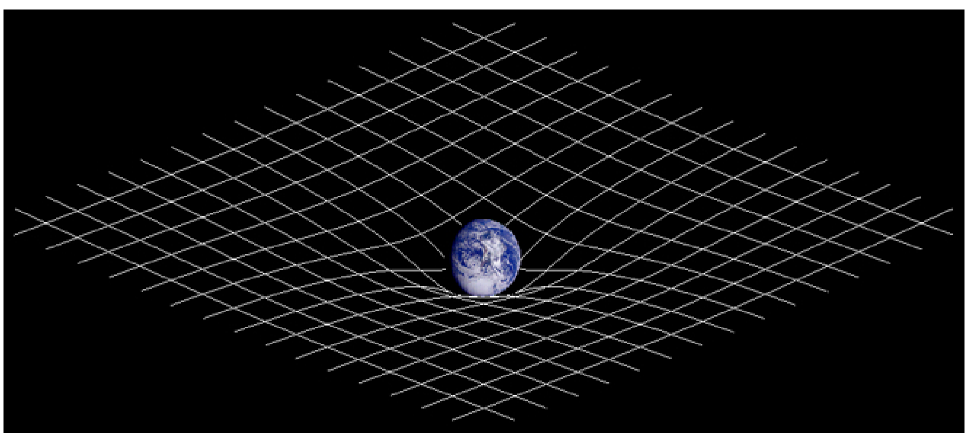
Oleg Gleizer LAMC

Introduction to Geometry Lesson 1

Geometry, the Art of Measuring Earth

Geometry is a composite word. In ancient Greek, *geo* meant *earth* and *metria* meant *measure*. Geometry was born as an art of measuring land lots. It progressed to become a science that explains, among other things, gravity as a curvature of warped spacetime.



The Earth bending space and time around it.

Father of Geometry

A Greek mathematician *Euclid* lived in the city of Alexandria, currently the second-largest city in Egypt.

Homework Problem 1 *Find Alexandria on the globe. Use Google Earth if you have none.*

Founded by *Alexander the Great* in 331 BC, it was one of more than twenty settlements the young conqueror named after himself. In his quest to take over the world, the Macedonian king was followed by a cohort of Greek scientists and philosophers, Euclid possibly being one of them.

After Alexander's sudden death, his empire broke into a bunch of warring principalities. Eventually *Ptolemy I*, one of the Alexander's most trusted and talented generals, emerged as a ruler of Egypt and founded the last dynasty of the Egyptian pharaohs, the *Ptolemaic Dynasty.* Euclid became famous during the reign of this king.

Euclid[1]

[1] This is an artist's rendering. No real portrait has reached our times.

We know very little about the man renown as the "Father of Geometry". His book, called "Elements", was the first geometry textbook on this planet. The most successful scientific publication of all times, it was in active use from the moment of its creation until the end of the 19th century, about 2,200 years altogether! Many of the results presented in the "Elements" were discovered by mathematicians preceding Euclid. It was his achievement to organize them in a logically coherent manuscript, including a system of rigorous proofs that remains the basis of mathematics 23 centuries later.

Common notions

A *definition* is a way to introduce a new object without ambiguity, using simpler, more fundamental notions to describe it. Note that we just have defined the word *definition*! Giving a clear, unambiguous definition is often not trivial.

Problem 1 *Define a chair.*

It is not possible to describe the most fundamental notions in Geometry using more fundamental notions. Therefore, the twenty three descriptions the "Elements" begin with are called *common notions*, not definitions. The first describes a point.

A point

Common Notion 1 *A point is that which has no part.*

Some ancient Greek philosophers believed that everything in nature was made of atoms, tiny indivisible parts of matter. The word *atom*, the anglicized *atomos*, actually meant *uncuttable* or *indivisible* in the language of Euclid and his contemporaries. Geometry constructed by the Greeks was a model world, a useful simulation of the more complicated world of physical reality they lived in. Some two millennia later, the real world is known to have 118 atoms, *basic chemical elements that cannot be broken further down by means of chemical reactions.* These atoms are not really indivisible. They are composed of the so-called elementary particles, protons, neutrons, and electrons. While the electrons are likely to be truly elementary, the protons and neutrons are made of the more fundamental bits of matter, called quarks.

Homework Problem 2 *What is the most abundant atom in our universe? Why?*

The notion of a point is so simple on the one hand, yet so deep on the other hand, that it cannot be explained using more fundamental notions. Nothing is more fundamental than a point, either in math or in physics.

To *prove* means to convince an educated sceptic by means of logic and reason.

Problem 2 *Prove that a point has no length, widths, or hight.*

A point and a dot

A *dot* is a model of a point that we draw on paper. Unlike a point, a dot has finite size.

Problem 3 *Draw a dot on the left-hand side of the space below. Imagine that you explore the drawing using some significant magnification, a powerful lens or a microscope. Draw the picture you would see on the right-hand side.*

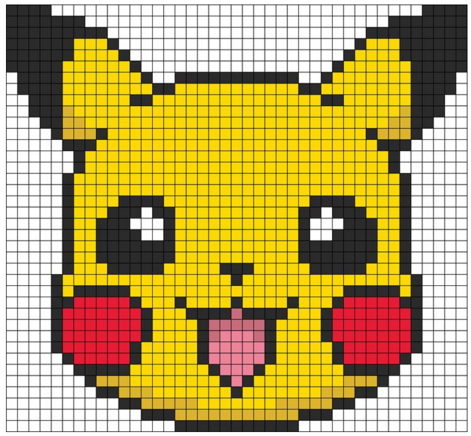
$$\xrightarrow{\textit{magnification}}$$

Problem 4 *Imagine that you look at an ideal point, an object with no length, height, or width, in a microscope. Would the picture change if you increase magnification?*

Since a point has no size in any direction, it is sometimes called a *zero-dimensional space*.

Pixels and dots

The word *pixel* stands for a *picture element*. A pixel is a luminous shape, a small circle, rectangle, wedge, etc., forming a picture on the screen of a computer, smartphone, tablet, TV, or other device. The color of a pixel is a mixture of three basic colors, red, green, and blue, all of varying degree of opacity. In Computer Science, they call the numerical value of the mix the RGB code of the color. For example, the RGB code of the yellow pixels below is (255,255,00) in the decimal system (and FFFF00 in the hexadecimal one).



Homework Problem 3 *What is the RGB code of the white color? Why?*

Homework Problem 4 *What is the RGB code of the black color? Why?*

The pixels of all the colors a video card of your device can produce are the atoms of the world you see on the screen of the device. Looking from afar, you do not see the pixels the on-screen picture is made of just like you do not see the atoms our world is made of.

Homework Problem 5 *How many pixels are there on the screen of your favorite device?*

A line

Common Notion 2 *A line is breadthless length.*

Note that Euclid calls a *line* what we typically call a *curve*. He does not mean that a line is necessarily a straight line. According to Euclid, a line (curve) has no height or width, only length. As any other geometric object, a line is made of points. Since, out of all dimensions, the lines defined above have only length, we call them one-dimensional (1D) objects.

Problem 5 *Draw a line on the left-hand side of the space below. Imagine that you explore the drawing using some significant magnification, a powerful lens or microscope. Draw the picture you would see on the right-hand side of the space below.*

$$\xrightarrow{\textit{magnification}}$$

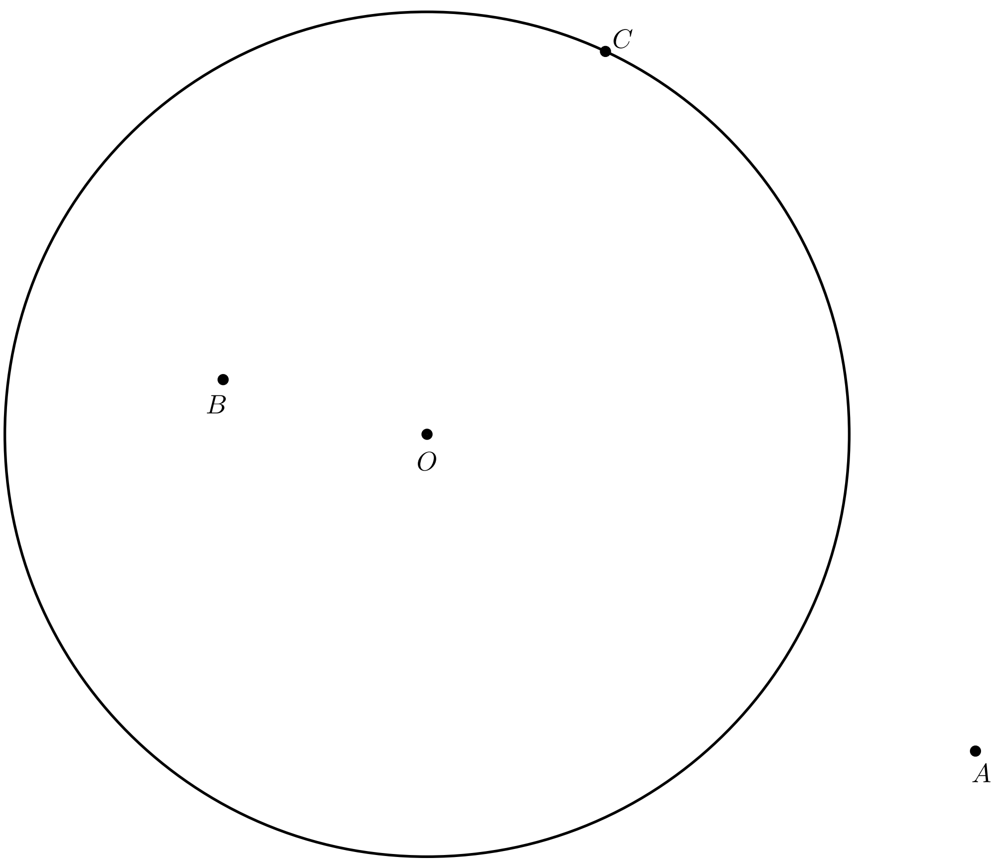
Problem 6 *Imagine that you look at an ideal line, an object having only length, but no height, or width, through a microscope. Would the picture change if you increase magnification? Why or why not?*

A circle and a disk

Definition 1 *A circle is the set of all the points in the plane having an equal distance, called the radius, from a special point, called the center.*

Problem 7 *What is a circle of radius zero?*

Problem 8 *Measure in inches the radius of the following circle. Round to the nearest inch.*



Measure the distances OA, OB, and OC with a ruler. Round to the nearest inch.

- *Which distance equals the radius? What point lies on the circle?*

- *Which distance is less than the radius? What point lies inside the circle?*

- *Which distance is greater than the radius? What point lies outside of the circle?*

- *Pick a point on the circle different from the point C. Tell the distance from the circle's center to the point without measuring it. Recall the definition of the circle if needed.*

- *Pick a point different from the point A outside of the circle. Is the distance from the circle's center to the point greater or less than the radius? Check the answer by a direct measurement, comparing the distance to the 2" mark on the ruler.*

- *Pick a point different from the point B inside the circle. Is the distance from the circle's center to the point greater or less than the radius? Check the answer by a direct measurement.*

Problem 9 *According to Euclid's Common Notion 2, is a circle a line? Why or why not?*

Definition 2 *A* <u>*disk*</u> *of radius r is the set of all the points in the plane such that their distances to a special point, called the center, are less than or equals to r.*

Problem 10 *What is the difference between a circle and a disk?*

Problem 11 *Take another look at the picture in Problem 8. Which of the points A, B, C, O belong to the circle? To the disk?*

The circle:

The disk:

Problem 12 *Suppose that you have to tend after a goat. You hammer a stake in the center of a flat meadow, tie up the goat to the stake with a rope of length* $r = 5$ *feet (5') and let it graze. What figure do you get when the animal eats up all the grass it can reach?*

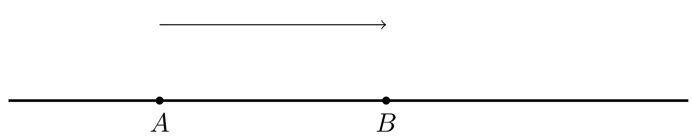
Homework Problem 6 *Given four stakes, a hammer, a collar, and an ample amount of rope, how do you make a goat eat the grass in the shape of a* $10' \times 20'$ *rectangle? (A goat can jump over a short fence.)*

Problem 13 *According to Definition 2, is a disk a line? Why or why not?*

A straight line

Common Notion 4 *A straight line is a line that lies evenly with the points on itself.*

What Euclid meant was that you can move any point of a straight line to the position of any other point by sliding the line along itself. Such a move is called a *translation*.

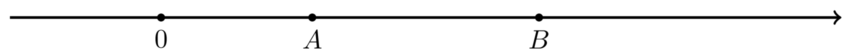


Definition 3 *A* <u>*symmetry*</u> *of a geometric object is a transformation that moves its points around, but preserves the object as a whole.*

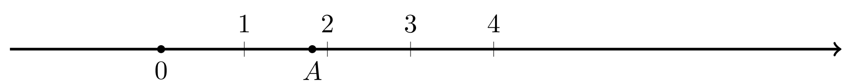
Euclid states that, unlike a generic line, a straight line is a highly symmetric object. Every translation of a straight line is a symmetry. It is this huge amount of symmetries that allows us to add and subtract numbers!

Addition as a symmetry

To explore the geometric nature of addition, let us pick up two points on the line, A and B.



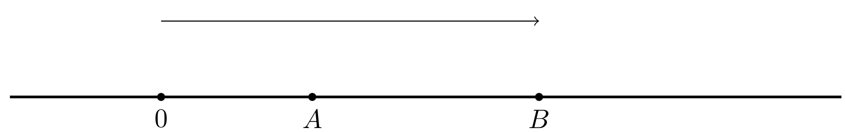
One can think of A as of the number equal to the distance between point A and zero. There exist two ways to measure the distance. The first is to take a ruler, a model of a straight line equipped with a unit step, 1 cm, 1 inch, 1 mile, or other and to count the number of the units tiling the segment from zero to A.



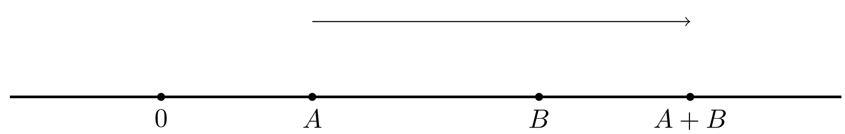
With this approach, the precision of the measurement is as good as the unit you choose. For more precision, one needs a smaller unit. The advantage of the method is that the point gets represented by a whole number or, at the very least, a mixed number that is not very complicated, like $1\frac{3}{4}$. The disadvantage is that the distance to nearly all the points of the line is measured approximately, not precisely. All you can say looking at the above picture is that A is somewhere in between 1.5 and 2, whereas in fact $A = 1\frac{9}{11}$.

Greek geometers mostly used a different approach. They measured the distance from zero to A not with a ruler, but with a compass, pretending that it was an infinitely precise instrument from the ideal world. This is enough to compare, add, and subtract numbers.

To add B to A in a geometric fashion, we need to invoke Euclid's Common Notion 4. Since all the points of a straight line lie evenly on the line, we can drag the line along itself until zero moves to the position of the point B.



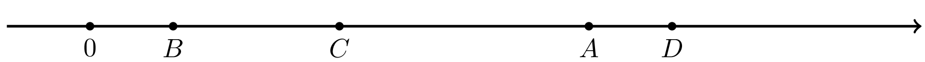
The last thing to do is to trace where the translation moves A. This point is the sum $A + B$.



The actual construction is implemented by measuring the distance from zero to B with a compass, then sticking the compass's needle at A and marking the point $A + B$ on the line.

Problem 14

• *Using a compass, mark the points $B + C$ and $D - A$ on the following number line.*



• *Put the correct sign, $>$, $<$, or $=$, in the box between the numbers.*

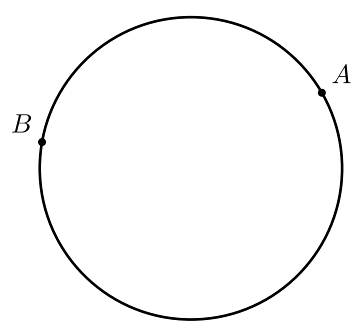
$$B + C \square A \qquad B \square 0 \qquad B \square D - A$$

• *Using a ruler marked with centimeters, find the numerical values of the numbers A, B, C, D. Round to the nearest centimeter. Then find $B + C$, and $D - A$ and check the correctness of the above geometric equalities and inequalities.*

$$A = \qquad B = \qquad C =$$

$$D = \qquad B + C = \qquad D \text{ - } A =$$

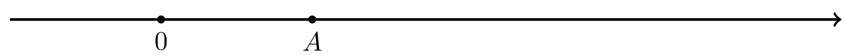
Problem 15 *Can we slide point A to the position of point B by a move that preserves the following line?*



Can we do it for any two points on a circumference?

Question 1 *It seems that, according to Euclid, a circumference is a straight line. How can we clarify the situation?*

Problem 16 *Use a compass to find the points* $2A$ *and* $3A$ *on the line below.*



Let us think of multiplication as of stretching the number line. This operation moves the points of the line, but preserves the line as a whole, so it also is a symmetry, called *dilation*. If we take both dilations and translations into account, then Euclid's Common Notion 4 describes nothing but a straight line!

Problem 17 *Imagine that you are an ancient Greek. You need to multiply some length by five. Can you do it using nothing but a rope? Draw a picture that shows how.*

Problem 18 *How many symmetries does a square have?*

Problem 19 *[1] A ruler has three marks, zero, seven, and eleven inches.*

- *Is it possible to use the ruler for drawing a segment that is 8" long?*

- *How about a 5"-long segment?*

Problem 20 *[1] Three houses A, B, and C, are built along a straight road.*

You are an engineer commissioned to find a place for a water well W so that the total distance from W to A, B, and C is the shortest possible. Where would you place the well?

Problem 21 *[1] This time they have four houses, A, B, C, and D, located along a straight road.*

Where would you build the well?

Problem 22 *Find the minimal value of the following sum. Hint: mark the points* 3*,* 7*, and* 8 *on the number line.*

$$|x-3|+|x-7|+|x-8|$$

Problem 23 *Find the minimal value of the following sum.*

$$|x+3|+|x|+|x-7|+|x-9|$$

Problem 24 *How many symmetries does a scalene triangle have? An equilateral triangle? Why?*

Self-test questions

- Why is Euclid called the Father of Geometry?

- What is a point?

- What is the original meaning of the word *atom*?

- How many different types of atoms are there in the real world?

- How many different types of atoms are there in the world of Geometry?

- What is a pixel?

- If you have to choose between the words a *point* and a *dot* to describe a pixel which one would you choose and why?

- Is every line straight?

- What is a line?

- What is a straight line?

- What is a symmetry?

- What symmetries of straight lines have we studied in the lesson?

- What is a circumference?

- What is a circle?

- Is a circumference a line?

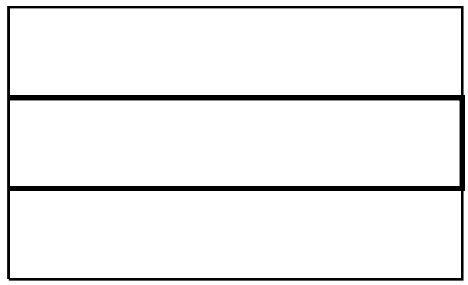
- What is the dimension of a circumference?

- Is a circle a line? Why or why not?

Additional reading: Space-filling curves

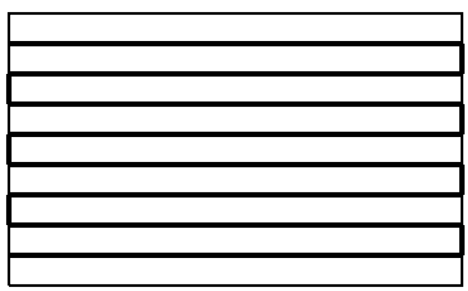
Below you will see something Euclid did not know, a curve of dimension two.

Let us take a rectangle, divide a pair of its opposite sides into three equal parts and draw the following line.



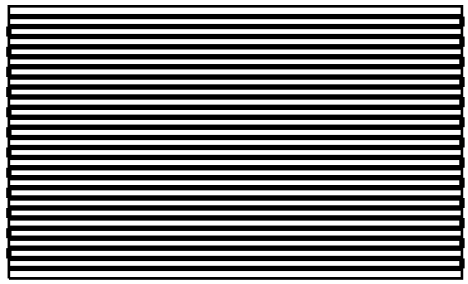
S_1

Let us call the line S_1, the letter S stemming from the word *snake*. (One can think of the above as of a picture of a snake taking a nap in a rectangular sandbox.) Let us divide the thirds of the original sides into three equal parts each and draw another snake, S_2.



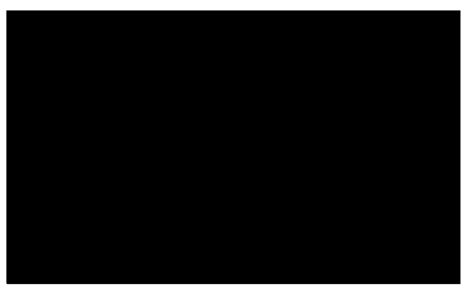
S_2

Let us continue the process to obtain S_3.



S_3

Let us keep dividing the segments into three parts and drawing the corresponding snake-lines. The line we get in the limit, S_∞,[2] will fill out the entire rectangle, intuitively a 2D shape!



S_∞

One can similarly construct a line filling out a 3D prism, or a higher dimensional solid. Lines having this property are called *space-filling curves.* Ancient Greeks were not aware of these monsters' existence. The lines Euclid considered were truly 1D in every meaning of the word.

[2]Reads as S-infinity.

References

[1] Alexander Shen, *Geometry in Problems*,
ISBN 978-1470419219