OLGA RADKO MATH CIRCLE, WINTER 2024: ADVANCED 3

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Worksheet 1:

Let $\mathbb{F}$ be a field. The *Projective Space* $\mathbb{P}^n_{\mathbb{F}}$ is defined to be the set of points with coordinates $[x_0 : x_1 : \ldots : x_n]$, with $x_i \in \mathbb{F}$ not all of them equal to 0, where two sets of coordinates $[x_0 : \ldots : x_n]$ and $[y_0 : \ldots y_n]$ define the same point if there exists a constant $\lambda \in \mathbb{F}$, such that for all i:

$$x_i = \lambda y_i.$$

For example, in $\mathbb{P}^2_{\mathbb{F}_3}$ the coordinates $[0 : 1 : 0]$ and $[0 : 2 : 0]$ define the same point.

Problem 1.0: Count how many points there are in the following projective spaces:

(1) $\mathbb{P}^2_{\mathbb{F}_2}$
(2) $\mathbb{P}^2_{\mathbb{F}_3}$
(3) $\mathbb{P}^3_{\mathbb{F}_2}$
(4) $\mathbb{P}^1_{\mathbb{F}_q}$

Solution 1.0:

Remember that *homogeneous polynomials of degree* d are polynomials whose nonzero terms are all of degree d.

The space $\mathbb{P}^2_{\mathbb{F}}$ is called the *projective plane over* $\mathbb{F}$ and the space $\mathbb{P}^1_{\mathbb{F}}$ is called the *projective line over* $\mathbb{F}$. *Lines* in a projective plane are defined by the zero-sets of homogeneous linear equations, i.e. the points $[x_0 : x_1 : x_2]$ of a line are the solutions of an equation:

$$ax_0 + bx_1 + cx_2 = 0.$$

Where a, b, c are in $\mathbb{F}$

An example of a line in $\mathbb{P}^2_{\mathbb{F}_3}$ are the points that satisfy the equation $x_1 + 2x_2 = 0$, which have coordinates $\{[1 : 1 : 0], [1 : 1 : 1], [1 : 1 : 2], [0 : 0 : 1]\}$.

Problem 1.1: For the following lines write down the list of their points :

- $x_0 + x_1 + x_2 = 0$ in $\mathbb{P}^2_{\mathbb{F}_2}$
- $x_0 + x_1 + x_2 = 0$ in $\mathbb{P}^2_{\mathbb{F}_3}$
- $x_0 + x_1 + x_2 = 0$ in $\mathbb{P}^2_{\mathbb{F}_5}$

How many points does a line in $\mathbb{P}^2_{\mathbb{F}_q}$ have?

Solution 1.1:

Problem 1.2: How many different points are there in $\mathbb{P}^2_{\mathbb{F}_q}$?

How many different lines are there in $\mathbb{P}^2_{\mathbb{F}_q}$?

Solution 1.2:

Problem 1.3: Given one fixed point in $\mathbb{P}^2_{\mathbb{F}_5}$. How many different lines pass through it?

How many lines pass through a fixed point in $\mathbb{P}^2_{\mathbb{F}_q}$?

Solution 1.3:

Two different lines in $\mathbb{P}^2_{\mathbb{F}_q}$ always intersect in exactly one point and given two points exactly one line passes through them. You may use this fact freely.

A line arrangement in $\mathbb{P}^2_{\mathbb{F}}$ is a (finite) set of lines and the set of points of intersection of them.

Problem 1.4: Can you find line arrangements with exactly $0, 1, 2$ and 3 points?

Can you classify the possible line arrangements with at most 3 points?

Solution 1.4:

Problem 1.5: Can you find different line arrangements where there are no points contained in exactly two lines? Can you do so in $\mathbb{P}^2_{\mathbb{R}}$?
Solution 1.5:

In general a *curve* in $\mathbb{P}^2_{\mathbb{F}_q}$ is given by the zero-set of a homogeneous polynomial.

The *degree* of a curve is the minimal d such that the curve is the zero-set of a homogeneous polynomial of degree d.

Problem 1.6: A line could also be given as the zero-set of a homogeneous polynomial of degree larger than one, can you give an example of this phenomenon?

Do all degree d curves in $\mathbb{P}^2_{\mathbb{F}_q}$ have the same number of points?

Solution 1.6:

The projective plane $\mathbb{P}^2_{\mathbb{F}}$ can be split into a copy of $\mathbb{F}^2$ given by the points with coordinates $[x_0 : x_1 : 1]$ and a projective line $\mathbb{P}^1_{\mathbb{F}}$ given by the points with coordinates $[x_0 : x_1 : 0]$.

Similarly a projective line $\mathbb{P}^1_{\mathbb{F}}$ can be split into a copy of $\mathbb{F}^1$ given by the points with coordinates $[x_0 : 1]$ and a points with coordinate $[1 : 0]$

Problem 1.7:

- Explain why this covers all the possible points of $\mathbb{P}^2_{\mathbb{F}}$
- Apply these same logic to decompose $\mathbb{P}^n_{\mathbb{F}}$ into affine spaces (i.e. spaces of the form $\mathbb{F}^i$).
- How many points are there in the space $\mathbb{P}^n_{\mathbb{F}_q}$?

Solution 1.7:

Problem 1.8: Can line arrangements have fewer points than lines?

Can a line arrangement with at least two points have fewer points than lines?

Can you classify all the line arrangements with fewer points than lines? **Solution 1.8:**

Problem 1.9: Can you classify all the line arrangements with the same number of lines than points?
Solution 1.9:
Problem 1.10: Let p and q be different prime numbers. Prove that the line arrangement $\mathcal{L}$ constructed by all the lines in $\mathbb{P}^2_{\mathbb{F}_p}$ does not exist in $\mathbb{P}^2_{\mathbb{F}_q}$, i.e. for any line arrangement $\mathcal{K}$ in $\mathbb{P}^2_{\mathbb{F}_q}$ there does not exist a bijection between $\mathcal{L}$ and $\mathcal{K}$ sending lines to lines and points to points, respecting the inclusions.
Solution 1.10:

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