Worksheet 1:

Let $F$ be a field. The Projective Space $\mathbb{P}^n_F$ is defined to be the set of points with coordinates $[x_0 : x_1 : \ldots : x_n]$, with $x_i \in F$ not all of them equal to 0, where two sets of coordinates $[x_0 : \ldots : x_n]$ and $[y_0 : \ldots : y_n]$ define the same point if there exists a constant $\lambda \in F$, such that for all $i$:

$$x_i = \lambda y_i.$$ 

For example, in $\mathbb{P}_2^2$, the coordinates $[0 : 1 : 0]$ and $[0 : 2 : 0]$ define the same point.

**Problem 1.0:** Count how many points there are in the following projective spaces:

1. $\mathbb{P}_2^2_F$
2. $\mathbb{P}_3^2_F$
3. $\mathbb{P}_2^3_F$
4. $\mathbb{P}_q^1_F$

**Solution 1.0:**
Remember that homogeneous polynomials of degree \(d\) are polynomials whose nonzero terms are all of degree \(d\).

The space \(\mathbb{P}_d^2\) is called the projective plane over \(\mathbb{F}\) and the space \(\mathbb{P}_1^1\) is called the projective line over \(\mathbb{F}\). Lines in a projective plane are defined by the zero-sets of homogeneous linear equations, i.e. the points \([x_0 : x_1 : x_2]\) of a line are the solutions of an equation:

\[ax_0 + bx_1 + cx_2 = 0.\]

Where \(a, b, c\) are in \(\mathbb{F}\)

An example of a line in \(\mathbb{P}_2^2\) are the points that satisfy the equation \(x_1 + 2x_2 = 0\), which have coordinates \([1 : 1 : 0], [1 : 1 : 1], [1 : 1 : 2], [0 : 0 : 1]\).

**Problem 1.1:** For the following lines write down the list of their points:

- \(x_0 + x_1 + x_2 = 0\) in \(\mathbb{P}_2^2\)
- \(x_0 + x_1 + x_2 = 0\) in \(\mathbb{P}_3^3\)
- \(x_0 + x_1 + x_2 = 0\) in \(\mathbb{P}_5^5\)

How many points does a line in \(\mathbb{P}_q^2\) have?

**Solution 1.1:**
Problem 1.2: How many different points are there in $\mathbb{P}_q^2$? How many different lines are there in $\mathbb{P}_q^2$? 

Solution 1.2:
Problem 1.3: Given one fixed point in $\mathbb{P}^2_{\mathbb{F}_q}$. How many different lines pass through it?

How many lines pass through a fixed point in $\mathbb{P}^2_{\mathbb{F}_q}$?

Solution 1.3:
Two different lines in $\mathbb{P}^2_F$ always intersect in exactly one point and given two points exactly one line passes through them. You may use this fact freely.

A line arrangement in $\mathbb{P}^2_F$ is a (finite) set of lines and the set of points of intersection of them.

**Problem 1.4:** Can you find line arrangements with exactly 0, 1, 2 and 3 points?

Can you classify the possible line arrangements with at most 3 points?

**Solution 1.4:**
Problem 1.5: Can you find different line arrangements where there are no points contained in exactly two lines? Can you do so in $\mathbb{P}^2_\mathbb{R}$?

Solution 1.5:
In general a curve in $\mathbb{P}^2_{\mathbb{F}_q}$ is given by the zero-set of a homogeneous polynomial.

The degree of a curve is the minimal $d$ such that the curve is the zero-set of a homogeneous polynomial of degree $d$.

**Problem 1.6:** A line could also be given as the zero-set of a homogeneous polynomial of degree larger than one, can you give an example of this phenomenon?

Do all degree $d$ curves in $\mathbb{P}^2_{\mathbb{F}_q}$ have the same number of points?

**Solution 1.6:**
The projective plane \( \mathbb{P}^2_F \) can be split into a copy of \( \mathbb{F}^2 \) given by the points with coordinates \([x_0 : x_1 : 1]\) and a projective line \( \mathbb{P}^1_F \) given by the points with coordinates \([x_0 : x_1 : 0]\).

Similarly a projective line \( \mathbb{P}^1_F \) can be split into a copy of \( \mathbb{F}^1 \) given by the points with coordinates \([x_0 : 1]\) and a points with coordinate \([1 : 0]\)

**Problem 1.7:**

- Explain why this covers all the possible points of \( \mathbb{P}^2_F \)
- Apply these same logic to decompose \( \mathbb{P}^2_F \) into affine spaces (i.e. spaces of the form \( \mathbb{F}^n \)).
- How many points are there in the space \( \mathbb{P}^n_{F_q} \)?

**Solution 1.7:**
**Problem 1.8:** Can line arrangements have fewer points than lines?
Can a line arrangement with at least two points have fewer points than lines?
Can you classify all the line arrangements with fewer points than lines? **Solution 1.8:**
**Problem 1.9:** Can you classify all the line arrangements with the same number of lines than points?

**Solution 1.9:**

**Problem 1.10:** Let $p$ and $q$ be different prime numbers. Prove that the line arrangement $\mathcal{L}$ constructed by all the lines in $\mathbb{P}_F^2$ does not exist in $\mathbb{P}_F^2$, i.e. for any line arrangement $\mathcal{K}$ in $\mathbb{P}_F^2$ there does not exist a bijection between $\mathcal{L}$ and $\mathcal{K}$ sending lines to lines and points to points, respecting the inclusions.

**Solution 1.10:**